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Math 255
HW # 7

Excellent! 2/2

8.1

6) Verify Green's theorem for $D=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$, $P(x,y)=\sin x$, $Q(x,y)=\cos y$

$$\int_{\partial D} P\,dx+Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{\partial \cos y}{\partial x}-\frac{\partial \sin x}{\partial y}\right)dx\,dy$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}(0-0)\,dx\,dy = \boxed{0}$$

12) Using the divergence theorem, show $\int_{\partial D}\vec{F}\cdot\vec{n}\,ds=0$ where $\vec{F}(x,y)=(y,-x)$
D: unit disc

$$\int_{\partial D}\vec{F}\cdot\vec{n}\,ds = \iint_D(\nabla\cdot\vec{F})\,dA = \iint_D\left(\frac{\partial y}{\partial x}+\frac{\partial(-x)}{\partial y}\right)dA = \iint_D(0+0)\,dA = \boxed{0}$$

14) Under the conditions of Green's theorem, Prove:

a) $$\int_{\partial D}PQ\,dx+PQ\,dy = \iint_D\left[Q\left(\frac{\partial P}{\partial x}-\frac{\partial P}{\partial y}\right)+P\left(\frac{\partial Q}{\partial x}-\frac{\partial Q}{\partial y}\right)\right]dx\,dy$$

Green's theorem says: $\int_{\partial D}P\,dx+Q\,dy = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$

Then, $$\int_{\partial D}PQ\,dx+PQ\,dy = \iint_D\left(\frac{\partial(PQ)}{\partial x}-\frac{\partial(PQ)}{\partial y}\right)dx\,dy = \iint_D\left[\left(Q\frac{\partial P}{\partial x}+P\frac{\partial Q}{\partial x}\right)-\left(Q\frac{\partial P}{\partial y}+P\frac{\partial Q}{\partial y}\right)\right]dx\,dy$$

$$=\boxed{\iint_D\left[Q\left(\frac{\partial P}{\partial x}-\frac{\partial P}{\partial y}\right)+P\left(\frac{\partial Q}{\partial x}-\frac{\partial Q}{\partial y}\right)\right]dx\,dy}$$

b) $$\int_{\partial D}\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)dx+\left(P\frac{\partial Q}{\partial y}-Q\frac{\partial P}{\partial y}\right)dy = 2\iint_D\left(P\frac{\partial^2 Q}{\partial x\partial y}-Q\frac{\partial^2 P}{\partial x\partial y}\right)dx\,dy$$

Based on Green's theorem, $\int_{\partial D}\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)dx+\left(P\frac{\partial Q}{\partial y}-Q\frac{\partial P}{\partial y}\right)dy$

$$=\iint_D\left(\frac{\partial\left(P\frac{\partial Q}{\partial y}-Q\frac{\partial P}{\partial y}\right)}{\partial x}-\frac{\partial\left(Q\frac{\partial P}{\partial x}-P\frac{\partial Q}{\partial x}\right)}{\partial y}\right)dx\,dy = \iint_D\left(\frac{\partial\left(P\frac{\partial Q}{\partial y}-Q\frac{\partial P}{\partial y}\right)}{\partial x}+\frac{\partial\left(P\frac{\partial Q}{\partial x}-Q\frac{\partial P}{\partial x}\right)}{\partial y}\right)dx\,dy$$

$$=\iint_D\left[\left(P\frac{\partial^2 Q}{\partial x\partial y}-Q\frac{\partial^2 P}{\partial x\partial y}\right)+\left(P\frac{\partial^2 Q}{\partial x\partial y}-Q\frac{\partial^2 P}{\partial x\partial y}\right)\right]dx\,dy = \boxed{2\iint_D\left(P\frac{\partial^2 Q}{\partial x\partial y}-Q\frac{\partial^2 P}{\partial x\partial y}\right)dx\,dy}$$

15) Evaluate the line integral $\int_C (2x^3-y^3)\,dx + (x^3+y^3)\,dy$, C: unit Circle and verify Green's theorem.

let $(x,y) \mapsto (\cos t, \sin t)$ for $t \in [0, 2\pi]$

then, $\int_C (2x^3-y^3)\,dx + (x^3+y^3)\,dy = \int_0^{2\pi} \big((2\cos^3 t - \sin^3 t)(-\sin t) + (\cos^3 t + \sin^3 t)(\cos t)\big)dt$

$$= \int_0^{2\pi} \big[(-2\cos^3 t \sin t + \sin^4 t) + (\cos^4 t + \sin^3 t \cos t)\big]\,dt$$

$$= \int_0^{2\pi} \big[\sin^4 t + \cos^4 t + (\sin t \cos t)(\sin^2 t - 2\cos^2 t)\big]\,dt$$

$$= \int_0^{2\pi} \big[(1 - 2\sin^2 t\cos^2 t) + (\sin t \cos t)(1 - 3\cos^2 t)\big]dt = \int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt - \frac{1}{2}\int_0^{2\pi} (3\cos^2 t - 1)\sin 2t\,dt$$

$$= \int_0^{2\pi} dt - \frac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt - \int_0^{2\pi} \big((-2\sin t)(3\sin^2 t - 2)\big)\cos t\,dt \qquad \int_0^{2\pi} dt = t\Big|_0^{2\pi}$$

$$\frac{1}{2}\int_0^{2\pi} \sin^2 2t\,dt = \begin{matrix} u = 2t \\ du = 2dx \end{matrix} \Rightarrow \frac{1}{2}\int \sin^2 u\,du = \frac{1}{2}\left(\frac{1}{2}\int du - \frac{\cos u \sin u}{2}\right) = \frac{u}{4} - \frac{\cos u \sin u}{4} \Rightarrow \frac{t}{4} - \frac{\cos(2t)\sin(2t)}{8}\Bigg|_0^{2\pi}$$

$$\frac{1}{2}\int_0^{2\pi} \cos t(-2\sin t(3\sin^2 t - 2))\,dt = \begin{matrix} u = \sin t \\ du = \cos t\,dt \end{matrix} \Rightarrow -2\int u(3u^2-2)\,du \quad \begin{matrix} v = 3u^2 - 2 \\ dv = 6u\,du \end{matrix} \Rightarrow -2\left(\frac{1}{6}\int v\,dv\right)$$

$$= -2\left(\frac{1}{6}\left(\frac{v^2}{2}\right)\right) = -2\left(\frac{v^2}{12}\right) = -\frac{v^2}{6} = \frac{-1}{2}\left(\frac{(3u^2-2)^2}{6}\right) = \frac{(3\sin^2 t - 2)^2}{12}\Bigg|_0^{2\pi}$$

$$\Rightarrow \frac{(3\sin^2 t - 2)^2}{12}\Bigg|_0^{2\pi} + \left(\frac{\cos(2t)\sin(2t)}{8} - \frac{t}{4}\right)\Bigg|_0^{2\pi} + t\Big|_0^{2\pi}$$

$$\left(-\frac{2}{12} + \frac{2}{12}\right) + \left(\left(0 - \frac{2\pi}{4}\right) - (0+0)\right) + 2\pi = 2\pi - \frac{2\pi}{4} = \frac{8\pi - 2\pi}{4} = \frac{6\pi}{4} = \boxed{\frac{3\pi}{2}}$$

<u>Using Green's theorem</u>: $P = 2x^3 - y^3 \quad Q = x^3 + y^3$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \iint_D \frac{\partial(x^3+y^3)}{\partial x} - \frac{\partial(2x^3 - y^3)}{\partial y}\,dx\,dy = \iint_D (3x^2 + 3y^2)\,dx\,dy$$

$$= 3\iint_D (x^2+y^2)\,dx\,dy \qquad \text{let } r = x+y,\ r^2 = x^2+y^2,\ r \in [0,1] \quad \theta \in [0, 2\pi]$$

$$\Rightarrow 3\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = 3\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = 3\int_0^{2\pi} \frac{1}{4}\,d\theta = 3\left(\frac{2\pi}{4}\right) = \frac{6\pi}{4} = \boxed{\frac{3\pi}{2}}$$

Thus, Green's theorem is verified.

19) a) verify the divergence theorem for $\vec{F}=(x,y)$, D: unit disc $x^2+y^2 \leq 1$

Proof: let $(x,y) \rightsquigarrow (\cos t, \sin t)$ for $t\in[0,2\pi]$

$$\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} (\cos t+\sin t)\cdot\vec{n}\,dt \qquad \vec{n} = \frac{(\cos t+\sin t)}{\sqrt{\cos^2 t+\sin^2 t}} = (\cos t, \sin t)$$

$$= \int_0^{2\pi} (\cos t+\sin t)\cdot(\cos t+\sin t)\,dt = \int_0^{2\pi} (\cos^2 t+\sin^2 t)\,dt = \int_0^{2\pi} dt = \underline{2\pi}$$

The divergence theorem says, $\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D (\nabla\cdot\vec{F})\,dA$

$$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1$$

$$\Rightarrow \iint_D (\nabla\cdot\vec{F})\,dA = \int_0^1\int_0^{2\pi} dA = \underline{2\pi}$$

The same result was obtained through the divergence theorem, thus, $\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D (\nabla\cdot\vec{F})\,dA$ and the divergence theorem is correct. □

b) Evaluate the integral of the normal component of $\vec{F}=(2xy - y^2)$

D: ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Based on the divergence theorem, $\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = \iint_D (\nabla\cdot\vec{F})\,dA$

$$(\nabla\cdot\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial(2xy)}{\partial x} + \frac{\partial(-y^2)}{\partial y} = 2y + (-2y) = 0$$

$$\Rightarrow \iint_D (\nabla\cdot\vec{F})\,dA = \iint_D 0\,dA = 0 \Rightarrow \boxed{\int_{\partial D} \vec{F}\cdot\vec{n}\,ds = 0}$$

20) Let $P(x,y) = \frac{-y}{(x^2+y^2)}$ and $Q(x,y) = \frac{x}{(x^2+y^2)}$, D: Unit disc

Why does Green's theorem fail for this P and Q?

These choices of P and Q will fail because, Green's theorem requires the functions P and Q to be C'-functions. This means they are differentiable and continuous however, P and Q are not continuous at $x=0$ and $y=0$. Thus, Green's theorem is not applicable.

24) Use theorem 2 to recover the formula $A = \frac{1}{2}\int_a^b r^2 \, d\theta$ for a region in Polar coordinates.

Th. 2 says: C: Simple closed curve where Green's theorem applies

then, $A(D) = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$ where $C = \partial D$

let region D be a cylinder of radius r, $\theta \in [a,b]$

then, $x = r\cos\theta$, $y = r\sin\theta$; $\phi(r,\theta) = (r\cos\theta, r\sin\theta)$

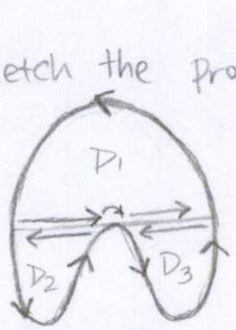
$dx = r\cos\theta - r\sin\theta \quad dy = r\sin\theta + r\cos\theta$

$$\Rightarrow \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_a^b \left(r\cos\theta(r\sin\theta + r\cos\theta) - r\sin\theta(r\cos\theta - r\sin\theta)\right) d\theta$$

$$= \frac{1}{2}\int_a^b \left(r^2\cos^2\theta + \cancel{r^2\cos\theta\sin\theta} - \cancel{r^2\sin\theta\cos\theta} + r^2\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_a^b \left(r^2\cos^2\theta + r^2\sin^2\theta\right) d\theta = \boxed{\frac{1}{2}\int_a^b r^2 \, d\theta}$$

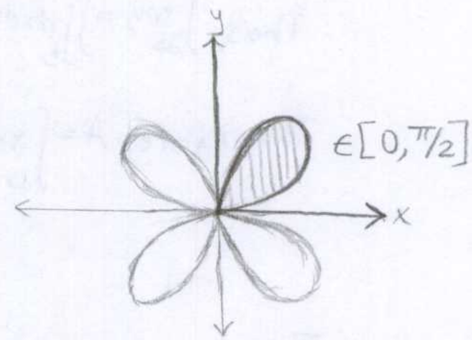
25) Sketch the proof of Green's theorem for region D:



Sketch of Proof: D is split into regions D_1, D_2, D_3 where Green's theorem applies.

Apply Green's theorem to each region and sum the results. Because the inner vectors point in opposite directions, they will cancel each other and only the value for the outer boundary will remain, this is the boundary for D. Thus, adding the values obtained for D_1, D_2, D_3 will result in the value of Green's theorem applied to D. □

27) Use Green's theorem to find the area of one loop of the four leafed rose $r = 3\sin 2\theta$ $\quad (x\,dy - y\,dx = r^2 d\theta)$



$$A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\int_a^b r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} (3\sin 2\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)\,d\theta \qquad u = 2\theta \quad du = \frac{1}{2}d\theta \Rightarrow \frac{9}{4}\int_0 \sin^2 u\,du$$

$$= \frac{9}{4}\int du - \frac{\cos u \sin u}{2} = \frac{9u}{4} - \frac{9(\cos u \sin u)}{8} \quad \text{Plug in } 2\theta = u$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)\,d\theta = \frac{9(2\theta)}{4} - \frac{9(\cos(2\theta)\sin(2\theta))}{8} = \frac{9(4\theta - \sin(4\theta))}{16}\Bigg|_0^{\pi/2}$$

$$= \left[\left(\frac{9(2\pi)}{16} - \cancel{\frac{9(\sin 2\pi)}{16}}^{\,0}\right) - \left(\cancel{\frac{9(0)}{16}}^{\,0} - \cancel{\frac{9\sin(0)}{16}}^{\,0}\right)\right] = \frac{18\pi}{16} = \frac{9\pi}{8}$$

$$\Rightarrow \boxed{A = \frac{9\pi}{8}}$$

28) Show that if C: simple closed curve where Green's theorem applies, then A(D) where $C = \partial D \Rightarrow A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx$ Show how this implies theorem 2.

Green's theorem says: $\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Proving $: A = \int_{\partial D} x\,dy = \int_{\partial D} -y\,dx$: LHS = let $P = 0,\ Q = x \Rightarrow \int_{\partial D} P\,dx + Q\,dy = \int_{\partial D} x\,dy = \iint_D \frac{\partial x}{\partial x}\,dx\,dy$

$$= \iint_D dx\,dy = A$$

RHS = let $P=-y,\ Q=0 \Rightarrow \int_{\partial D} P\,dx + Q\,dy = \int_{\partial D} -y\,dx = \iint_D \left(0 - \frac{\partial(-y)}{\partial y}\right) dx\,dy = \iint_D dx\,dy = A$

Thus, $\int_{\partial D} x\,dy = \iint_D dx\,dy = A = \iint_D dx\,dy = \int_{\partial D} -y\,dx$

Therefore, $A = \int_{\partial D} x\,dy = \int_{\partial D} -y\,dx$.

Theorem 2 says: $A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$

It is given that $A = \int_{\partial D} x\,dy = \int_{\partial D} -y\,dx$. If $\int_{\partial D} x\,dy + \int_{\partial D} -y\,dx$ then,

$\int_{\partial D} x\,dy - y\,dx = 2A \Rightarrow A = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$

Thus, $A = \int_{\partial D} x\,dy = \int_{\partial y} -y\,dx$ implies theorem 2.