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MATH-255
04/21/17

2/2 Excellent!

$\boxed{8.2}$ - #4, 12, 15, 16, 24, 26, 29, 31

4)
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = (0-1)\hat{i} - (1-0)\hat{j} + (0-1)\hat{k} = -\hat{i} - \hat{j} - \hat{k}$$

$z = 5-2x-3y \Rightarrow \Phi(u,v) = (u, v, 5-2u-3v)$ is the param. of S.
$-1 \le u \le 2, \quad 1 \le v \le 3$

∂S can be parametrized as union of 4 pieces of curves:

$(-1,1,4)$ to $(2,1,-2)$: C_1: $(-1,1,4) + t(2-(-1), 1-1, -2-4) = (3t-1, 1, -6t+4)$, $t \in [0,1]$

$(2,1,-2)$ to $(2,3,-8)$ C_2: $(2,1,-2) + t(0,2,-6) = (2, 2t+1, -6t-2)$, $t \in [0,1]$

$(2,3,-8)$ to $(-1,3,2)$ C_3: $(2,3,-8) + t(-3,0,6) = (-3t+2, 3, 6t-8)$, $t \in [0,1]$

$(-1,3,2)$ to $(-1,1,4)$ C_4: $(-1,3,2) + t(0,-2,6) = (-1, -2t+3, 6t-2)$, $t \in [0,1]$

- $$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_{0, C_1}^{1} (1, -6t+4, 3t-1) \cdot (3, 0, -6)\, dt + \int_{0, C_2}^{1} (2t+1, -6t-2, 2) \cdot (0, 2, -6)\, dt$$
$$+ \int_{0, C_3}^{1} (3, 6t-8, -3t+2) \cdot (-3, 0, 6)\, dt + \int_{0, C_4}^{1} (-2t+3, 6t-2, -1) \cdot (0, -2, 6)\, dt$$
$$= \int_0^1 3 + -18t + 6\, dt + \int_0^1 -12t - 4 - 12\, dt + \int_0^1 -9 - 18t + 12\, dt + \int_0^1 -12t + 4 - 6\, dt$$
$$= \int_0^1 -18t + 9\, dt + \int_0^1 -12t - 16\, dt + \int_0^1 -18t + 3\, dt + \int_0^1 -12t - 2\, dt$$
$$= -9t^2 + 9t \Big|_0^1 + -6t^2 - 16t \Big|_0^1 + -9t^2 + 3t \Big|_0^1 + -6t^2 - 2t \Big|_0^1$$
$$= \underline{-36}$$

- $$\iint_S (-1,-1,-1) \cdot dS = \int_1^3 \int_{-1}^2 \left(\frac{\partial z}{\partial u}\right) + \left(\frac{\partial z}{\partial v}\right) - 1 \; du\,dv = \int_1^3 \int_{-1}^2 -2 - 3 - 1\, du\,dv$$
$$= -6 \int_1^3 \int_{-1}^2 du\,dv = -6(3)(2) = \underline{-36}$$

12) $\int_{\partial S} \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot d\vec{S}$

Since $\vec{E}$ is normal to ∂S, it does 0 work:

$$\int_{\partial S} \vec{E} \cdot d\vec{s} = 0$$

$$\hookrightarrow -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot d\vec{S} = 0$$

$$\iint_S \vec{H} \cdot d\vec{S} = 0$$

Since zero is a constant, magnetic flux across S is constant.

15) $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$

$$\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = (y-z)\hat{i} - (x-z)\hat{j} + (x-y)\hat{k}$$

$\hookrightarrow \|\vec{F}\|$ is area spanned by (x,y,z) and $(1,1,1)$

$$\cancel{\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y-z & z-x & x-y \end{vmatrix} = (-1-1)\hat{i} - (1+1)\hat{j} + (-1-1)\hat{k} = (-2,-2,-2)}$$

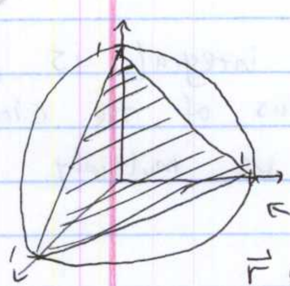
$$\cancel{x^2+y^2+z^2 = 1^2 \Rightarrow (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \Phi(\theta,\phi),\ 0}$$

$$\cancel{\vec{T}_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)}$$

$$\cancel{\vec{T}_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)}$$

$$\cancel{\vec{T}_\theta \times \vec{T}_\phi = -\sin^2\phi\cos\theta\,\hat{i} - (\sin^2\phi\sin\theta)\hat{j} + [-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi]\hat{k}}$$

$$\cancel{= (-\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi)}$$



$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot \frac{c'(t)}{\|c'(t)\|} \|c'(t)\| \, dt$$ call (*)

$\leftarrow \partial S$ is intersection, Since $\vec{F}$ is a cross prod. b/w $\vec{r}$ and $\hat{i}+\hat{j}+\hat{k} \Rightarrow (1,1,1)$, the points on ∂S imply $\vec{F}(x,y,z)$ is tangent vector to ∂S of length $\|\vec{F}\|$.

Using $(0,0,1)$, $\|\vec{F}\|$ at ∂S is $\sqrt{(-1)^2+(1)^2+0} = \sqrt{2}$

So, $(*) = \int_a^b \sqrt{2} \, \|c'(t)\| \, dt$, which is Simply,

$\sqrt{2} \cdot \int_a^b \|c'(t)\| \, dt$, which is the length of ∂S.

The length of ∂S is $\frac{\pi}{3}(2\sqrt{6})$, so

$$\sqrt{2} \cdot \frac{\pi}{3}(2\sqrt{6}) = \frac{2\pi\sqrt{12}}{3} = \frac{4\pi\sqrt{3}}{3}$$

16) From exercise 15, we got $\vec{F} = (y-z)\hat{i} + (z-x)\hat{j} + (x-y)\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y-z & z-x & x-y \end{vmatrix} = (-1-1)\hat{i} - (1+1)\hat{j} + (-1-1)\hat{k} = (-2,-2,-2)$$

$$\iint_S (-2,-2,-2) \cdot d\vec{s}$$

By Stokes' thm, since $\int_{\partial \Sigma} \vec{F} \cdot d\vec{r} = \iint_S (-2,-2,-2) \cdot d\vec{s}$,

all we have to do is find a right parametrization to compute $\iint_S (-2,-2,-2) \cdot d\vec{s}$, which will be the circular boundary.

~~So, our boundaries are $0 \le y \le x+1$ and $0 \le x \le 1$~~

Since $\|(-2,-2,-2)\|$ is $\sqrt{12}$, we know that our integral is $\sqrt{12}$ times the area of the circle. Since our radius of the circle is $\frac{\sqrt{2}}{\sqrt{3}}$, our area is $\frac{2}{3}\pi = r^2\pi$. So we multiply this area by $\sqrt{12}$ which is

$$\frac{2\sqrt{12}}{3}\pi = \frac{4\pi\sqrt{3}}{3}.$$

So it is same answer as #15, so we have computed our integral.

24) Using Stokes' thm,

$$\int_{\partial S} (\vec{v}\times\vec{r})\cdot d\vec{s} = \iint_S \left[\nabla\times(\vec{v}\times\vec{r})\right]\cdot d\vec{S}$$

$$\vec{v}\times\vec{r} = \left(v_2 z - v_3 y,\ -v_1 z + v_3 x,\ v_1 y - v_2 x\right)$$

$$\nabla\times(\vec{v}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ v_2 z - v_3 y & v_3 x - v_1 z & v_1 y - v_2 x \end{vmatrix} = (v_1+v_1)\hat{i} - (-v_2-v_2)\hat{j} + (v_3+v_3)\hat{k}$$

$$= 2v_1\hat{i} + 2v_2\hat{j} + 2v_3\hat{k}$$

$$= 2(v_1, v_2, v_3) = 2\vec{v}$$

$$\int_{\partial S} (\vec{v}\times\vec{r})\cdot d\vec{s} = \iint_S \left[\nabla\times(\vec{v}\times\vec{r})\right]\cdot d\vec{S}$$

$$= 2\iint_S \vec{v}\cdot d\vec{S}$$

$$= 2\iint_S \vec{v}\cdot\vec{n}\ dS$$

26a) By Stokes' thm, $\int_C f\nabla g \cdot d\vec{s} = \iint_S (\nabla \times f\nabla g) \cdot d\vec{S}$.

So, if we show $(\nabla \times f\nabla g) = (\nabla f \times \nabla g)$, we have established the equality.

$$\begin{aligned} \nabla \times f\nabla g &= f(\nabla \times \nabla g) + (\nabla f \times \nabla g) &&\text{, by prop. 10 (p.255)} \\ &= f(\vec{0}) + \nabla f \times \nabla g &&\text{, by prop. 11 (p.255)} \\ &= \nabla f \times \nabla g \end{aligned}$$

So, the two integrals are equivalent.

26b) $\int_C (f\nabla g + g\nabla f) \cdot d\vec{s} = 0 \Rightarrow \int_C f\nabla g \cdot d\vec{s} + \int_C g\nabla f \cdot d\vec{s}$

~~Notice that $f\nabla g$~~ By Stokes' thm this is simply,

$$\iint_S (\nabla \times f\nabla g) \cdot d\vec{S} + \iint_S (\nabla \times g\nabla f) \cdot d\vec{S}.$$

By part (a), this is equivalent to

$$\iint_S (\nabla f \times \nabla g) \cdot d\vec{S} + \iint_S (\nabla g \times \nabla f) \cdot d\vec{S}$$

By cross product property (p.37), $\nabla f \times \nabla g = -\nabla g \times \nabla f$
$= -(\nabla g \times \nabla f)$

So, $-\iint_S (\nabla g \times \nabla f) \cdot d\vec{S} + \iint_S (\nabla g \times \nabla f) \cdot d\vec{S} = 0$

29) $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} = (1)\hat{i} - (-1)\hat{j} + (1)\hat{k} = \hat{i} + \hat{j} + \hat{k}$$

$$\iint_S (1,1,1) \cdot d\vec{S} = \int_{\partial S} (z, x, y) \cdot d\vec{s}$$

$\vec{T}_r = (\cos\theta,\ \sin\theta,\ 0)$

$\vec{T}_\theta = (-r\sin\theta,\ r\cos\theta,\ 1)$

$\vec{T}_r \times \vec{T}_\theta = (\sin\theta)\hat{i} - (\cos\theta)\hat{j} + (r\cos^2\theta + r\sin^2\theta)\hat{k}$

$= (\sin\theta,\ -\cos\theta,\ r)$

$$\int_0^{\frac{\pi}{2}}\int_0^1 (1,1,1)\cdot(\sin\theta, -\cos\theta, r)\ dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 (\sin\theta - \cos\theta + r)\ dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[r\sin\theta - r\cos\theta + \frac{r^2}{2}\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \sin\theta - \cos\theta + \frac{1}{2}\ d\theta$$

$$= -\cos\theta - \sin\theta + \frac{\theta}{2}\Big|_0^{\frac{\pi}{2}} = \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \frac{\pi}{4}\right) - (-\cos 0 - \sin 0)$$

$$= \left(0 - 1 + \frac{\pi}{4}\right) - (-1 - 0)$$

$$= \frac{\pi}{4}$$

Now we need to verify $\int_{\partial S} (z, x, y)\cdot d\vec{s} = \frac{\pi}{4}$

$$\Rightarrow \int_{\partial S} (\theta, r\cos\theta, r\sin\theta)\cdot d\vec{s} = \int_0^{\frac{\pi}{2}} -\theta\sin\theta + \cos^2\theta + \sin\theta\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta + \sin\theta(-\theta + 1)\ d\theta$$

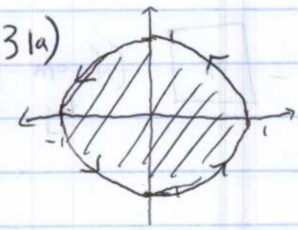
$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta + \int_0^{\frac{\pi}{2}} \sin\theta(-\theta+1)\ d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right]_0^{\frac{\pi}{2}} + \cancel{\left[(\theta - 1)\cos\theta - \sin\theta\right]_0^{\frac{\pi}{2}}}\ 0$$

$$= \frac{\pi}{4}$$

So, $\frac{\pi}{4} = \frac{\pi}{4}$ so Stokes' is verified. ✓

3.1a)



$$\vec{F} = (x^2,\ 2xy + x,\ z)$$

$$c(t) = (\cos t,\ \sin t,\ 0)$$

$$\iint_S \vec{F} \cdot d\vec{S} \quad , S: \Phi(r, \theta) = (r\cos\theta,\ r\sin\theta,\ 0), \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$\vec{T}_r = (\cos\theta,\ \sin\theta,\ 0)$$
$$\vec{T}_\theta = (-r\sin\theta,\ r\cos\theta,\ 0),$$

$$\vec{T}_r \times \vec{T}_\theta = (0)\hat{i} - (0)\hat{j} + (r\cos^2\theta + r\sin^2\theta)\hat{k} = (0, 0, r)$$

$$\vec{F}(\Phi(r,\theta)) = (r^2\cos^2\theta,\ 2r^2\cos\theta\sin\theta + r\cos\theta,\ 0)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (r^2\cos^2\theta,\ 2r^2\cos\theta\sin\theta + r\cos\theta,\ 0) \cdot (0, 0, r)\, dS$$

$$= \iint_S 0 \, dS = 0$$

3.1b)

$$\int_c \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{2\pi} (\cos^2 t,\ 2\cos t\sin t + \cos t,\ 0) \cdot (-\sin t,\ \cos t,\ 0)\, dt$$

$$= \int_0^{2\pi} \left(-\sin t\cos^2 t + 2\cos^2 t\sin t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(\cos^2 t\sin t + \cos^2 t\right) dt$$

We can use $d(\cos t)$ instead of dt to split the integral so that

$$= \int_0^{2\pi} \cos^2 t\ d(\cos t) + \int_0^{2\pi} \cos^2 t\, dt$$

$$= -\frac{\cos^3 t}{3}\Big|_0^{2\pi} + \frac{t + \sin t\cos t}{2}\Big|_0^{2\pi}$$

$$= 0 + \frac{2\pi}{2}$$

$$= \pi$$

31c) $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s} = \boxed{\pi}$ (from part b and by Stokes')

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2xy+x & z \end{vmatrix} = (0)\hat{i} - 0\hat{j} + (2y+1)\hat{k} = (0, 0, 2y+1)$$

$$\nabla \times \vec{F}(\Phi(r,\theta)) = (0, 0, 2r\sin\theta + 1)$$

$$\iint_S \nabla \times \vec{F}(\Phi(r,\theta)) \cdot d\vec{S} = \iint_S (0,0,2r\sin\theta + 1) \cdot \underbrace{(\vec{T}_r \times \vec{T}_\theta)}_{(0,0,r)} \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^2\sin\theta + r \, dr\, d\theta$$

$$= \cancel{\int_0^{2\pi} 2\sin\theta \int_0^1 r^2 + r} \int_0^{2\pi} \left[\frac{2}{3}r^3\sin\theta + \frac{1}{2}r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{2}{3}\sin\theta + \frac{1}{2}\right) d\theta$$

$$= -\frac{2}{3}\cos\theta + \frac{1}{2}\theta \Big|_0^{2\pi}$$

$$= \left(-\frac{2}{3} + \pi\right) - \left(-\frac{2}{3} + 0\right) = \boxed{\pi}$$

So Stokes' thm is verified. ✓