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HW 8: 8.2
MATH 25500
Professor Park

2/2 Excellent & Very good Work!

4. Verify Stokes' Theorem for the surface S, and $F = yi + zj + xk$. S is the portion of the plane $2x+3y+z=5$ lying between points $(-1,1,4)$, $(2,3,-8)$, and $(-1,3,2)$

$\Rightarrow z = 5-2x-3y$

$\Rightarrow \Phi: (u, v, 5-2u-3v),\ -1 \le u \le 2,\ 1 \le v \le 3$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y & z & x \end{vmatrix} = (-1,-1,-1)$$

$\vec{T}_u = (1,0,-2)$, $\vec{T}_v = (0,1,-3)$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{vmatrix} = (2,3,1)$$

$\partial S = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\left.\begin{aligned} C_1 &= (-1+3t,\ 1,\ 4-6t) \\ C_2 &= (2,\ 2t+1,\ -6t-2) \\ C_3 &= (-3t+2,\ 3,\ 6t-8) \\ C_4 &= (-1,\ -2t+3,\ 6t-2) \end{aligned}\right\} 0 \le t \le 1$$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iint_D (-1,-1,-1)\cdot(2,3,1)\,du\,dv = \int_{-1}^{2}\int_{1}^{3} -2-3-1\,du\,dv = \int_{-1}^{2}\int_{1}^{3} -6\,dv\,du$$

$$= \int_1^2 (-18+6)\,du = \int_1^2 -12\,du = -24 - 12 = -36.$$

12. Let S be a surface with boundary ∂S, and suppose E is an electric field that is perpendicular to ∂S. Show that the induced magnetic flux across S is constant in time. (HINT: use Faraday's Law)

induced magnetic flux across S is constant in time

$\Rightarrow$ derivative of magnetic flux with respect to time $= 0$

Since S is not a function of time,

$$\frac{\partial}{\partial t}\iint_S H \cdot dS = \iint_S \frac{\partial H}{\partial t}\,dS = -\iint_S -\frac{\partial H}{\partial t}\,dS$$

By Faraday's Law,

$$-\iint_S (\nabla \times E)\cdot dS$$

By Stokes' Theorem and the fact that $E \cdot ds = 0$ since E is perpendicular to the boundary of S,

$$-\int_{\partial S} E \cdot dS = 0.$$

15. Evaluate the integral $\iint_S (\nabla \times F)\cdot dS$, where S is the portion of the surface of a sphere defined by $x^2+y^2+z^2=1$ and $x+y+z \ge 1$, and where $F = r \times (i+j+k)$, $r = xi + yj + zk$.

By Stokes theorem, $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$.

Let $c(t)$ parameterize the curve ∂S for $t \in [a,b]$. Then,

$$\int_{\partial S} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot \left[\frac{c'(t)}{\|c'(t)\|}\right] \cdot \|c'(t)\|\,dt$$

Using geometry, ∂S is given by the intersection of the sphere $x^2+y^2+z^2=1$ and $x+y+z \ge 1$

$\vec{F} = r \times (i+j+k)$

$\Rightarrow$ For every point (x,y,z) on ∂S, $\vec{F}(x,y,z)$ is a vector tangent to ∂S of equal length to the area of the parallelogram spanned by (x,y,z) and $(1,1,1)$ pointing in the clockwise direction around ∂S. For every (x,y,z) on ∂S, the length of this vector is constant. Using the point $(x,y,z) = (1,0,0)$, the value is $\sqrt{2}$.

Using formula $ab = \|a\|\|b\|\cos\Theta$,

$$\int_a^b \vec{F} \cdot \left[\frac{c'(t)}{\|c'(t)\|}\right] \cdot \|c'(t)\|\,dt = -\int_a^b \sqrt{2}\,\|c'(t)\|\,dt$$

Since the angle between $c'(t)$ and $\vec{F}$ is π, hence the negative sign.

The value of $-\int_a^b \sqrt{2}\,\|c'(t)\|\,dt$ is $-\sqrt{2}\cdot$ arclength ∂S

arclength $\partial S = \frac{2\sqrt{6}\pi}{3}$

$\Rightarrow -\sqrt{2}\left[\frac{2\sqrt{6}\pi}{3}\right] = -\frac{4\sqrt{3}\pi}{3}$

16. Show that the calculation in 15 can be simplified by observing that $\int_{\partial S} F \cdot dr = \int_{\partial \Sigma} F \cdot dr$ for any other surface Σ. By picking Σ appropriately, $\iint_\Sigma (\nabla \times F)\cdot dS$ may be easy to compute. Show that this is the case if Σ is taken to be the portion of the plate $x+y+z=1$ inside the circle ∂S.

Man, You gotta do the work!!

24. For the surface S and a fixed vector v, prove that $2\iint_S v \cdot n\,dS = \int_{\partial S} (v \times r)\,ds$, where $r(x,y,z) = (x,y,z)$

Let $v = (a,b,c)$.

$$v \times r = \begin{vmatrix} i & j & k \\ a & b & c \\ x & y & z \end{vmatrix} = [(bz-cy), (cx-az), (ay-bx)]$$

$$\nabla \times (v \times r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz-cy & cx-az & ay-bx \end{vmatrix} = [(a+a), (b+b), (c+c)]$$

$$= (2a, 2b, 2c)$$

$$= 2v$$

$\therefore$ By Stokes' Theorem,

$$\int_{\partial S} (v \times r)\,dS = \iint_S (\nabla \times (v \times r))\,dS = 2\iint_S v \cdot n\,dS$$

26. If C is a closed curve that is the boundary of a surface S, and f and g are C^2 functions, show that

a) $\int_C f\nabla g \cdot dS = \iint_S (\nabla f \times \nabla g)\cdot dS$

Since C is a closed curve which is the boundary of surface S, $C = \partial S$. According to Stokes Theorem, $\iint_S (\nabla \times F)\cdot dS = \int_C F \cdot ds$

$\Rightarrow \nabla \times f\nabla g = \nabla f \times \nabla g$

From basic identity 10, $\mathrm{curl}(f\nabla g) = f\,\mathrm{curl}(\nabla g) + \nabla f \times \nabla g$

From basic identity 11, $\mathrm{curl}(\nabla g) = 0$

$\Rightarrow \mathrm{curl}(f\nabla g) = \nabla f \times \nabla g$

$\therefore \int_C f\nabla g \cdot dS = \iint_S (\nabla f \times \nabla g)\cdot dS$

b) $\int_C (f\nabla g + g\nabla f)\cdot dS = 0$

From basic identity 6, $\mathrm{curl}(f\nabla g + g\nabla f) = \mathrm{curl}(f\nabla g) + \mathrm{curl}(g\nabla f)$

From basic identity 10, $\mathrm{curl}(f\nabla g + g\nabla f) = f\,\mathrm{curl}(\nabla g) + \nabla f \times \nabla g + g\,\mathrm{curl}(\nabla f) + \nabla g \times \nabla f$

$= f\,\mathrm{curl}(\nabla g) + g\,\mathrm{curl}(\nabla f)$

From basic identity 11, $\mathrm{curl}(\nabla f) = \mathrm{curl}(\nabla g) = 0 \rightarrow \mathrm{curl}(f\nabla g + g\nabla f) = 0$

By Stokes' Theorem: $\int_C (f\nabla g + g\nabla f)\cdot dS = \iint_S \nabla \times (f\nabla g + g\nabla f)\cdot dS = \iint_S 0\,dS = 0$

29. Verify theorem 6 for the heliocoid $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$, $(r,\theta) \in [0,1]\times[0,\frac{\pi}{2}]$
and the vector field $F(x,y,z) = (z, x, y)$

$\Phi_r = (\cos\theta, \sin\theta, 0)$, $\Phi_\theta = (-r\sin\theta, r\cos\theta, 1)$

$$\Phi_r \times \Phi_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = [\sin\theta, -\cos\theta, (r\cos^2\theta - (-r\sin^2\theta))]$$

$$= (\sin\theta, -\cos\theta, r)$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1,1,1)$$

$$\iint_S (\nabla\times F)\cdot dS = \iint_S (1,1,1)\cdot(\sin\theta, -\cos\theta, r)\,dr\,d\theta$$

$$= \int_0^1\int_0^{\frac{\pi}{2}} (\sin\theta - \cos\theta + r)\,d\theta\,dr$$

$$= \int_0^1 \left[\frac{\pi}{2}r\right]dr$$

$$= \left[\frac{\pi r^2}{4}\right]_0^1$$

$$= \frac{\pi}{4}$$

∂S is composed of four parts:

$r=1$: $\Phi(1,\theta) = (\cos\theta, \sin\theta, \theta)$

$F = (\theta, \cos\theta, \sin\theta)$

$ds = (-\sin\theta, \cos\theta, 1)d\theta$

$$\hookrightarrow \int_{\partial S_1} F\cdot ds = \int_0^{\frac{\pi}{2}} (\theta, \cos\theta, \sin\theta)\cdot(-\sin\theta, \cos\theta, 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} [-\theta\sin\theta + \cos^2\theta + \sin\theta]\,d\theta$$

$$-\int\theta\sin\theta\,d\theta = -\int -\cos\theta\,d\theta - \theta\cos\theta$$

$$= -(\sin\theta - \theta\cos\theta)$$

$$= (\theta\cos\theta - \sin\theta)$$

$$\int\cos^2\theta\,d\theta = \int\frac{\cos(2\theta)+1}{2}\,d\theta$$

$$= \frac{1}{2}\int\cos(2\theta)\,d\theta + \frac{1}{2}\int d\theta$$

$u = 2\theta$, $du = 2d\theta$

$$\frac{1}{2}\int\cos(2\theta)\,d\theta = \frac{\sin 2\theta}{2}$$

$$\hookrightarrow \int\cos^2\theta\,d\theta = \frac{\sin(2\theta)}{4} + \frac{\theta}{2}$$

$$\hookrightarrow \int_0^{\frac{\pi}{2}} [-\theta\sin\theta + \cos^2\theta + \sin\theta]\,d\theta = \left[(\theta\cos\theta - \sin\theta) + \left(\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right) - \cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

$\theta = \pi/2$: $0 \le r \le 1$

$$\hookrightarrow \int_{\partial S_2} F\cdot ds = \int_1^0 (\tfrac{\pi}{2}, 0, r)\cdot(0,1,0)\,dr$$

$$= \int_1^0 (0+0+0)\,dr$$

$$= 0$$

$r=0$: $0 \le \theta \le \frac{\pi}{2}$

$$\hookrightarrow \int_{\partial S_3} F\cdot ds = \int_{\frac{\pi}{2}}^0 (\theta, 0, 0)\cdot(0,0,1)\,d\theta$$

$$= \int_{\frac{\pi}{2}}^0 (0+0+0)\,d\theta$$

$$= 0$$

$\theta = 0$: $\int_{\partial S_4} F\cdot ds = \int_0^1 (0, r, 0)\cdot(1,0,0)\,dr$

$$= 0$$

$$\int_{\partial S} F\cdot ds = \frac{\pi}{4} + 0 + 0 + 0$$

$$= \frac{\pi}{4}$$ (theorem 6 is verified)

31. Let $F = x^2 i + (2xy + x)j + zk$. Let C be the circle $x^2+y^2=1$ and let S be the disc $x^2+y^2 \le 1$ within the plane $z=0$

a) Determine the flux out of S.

$C: x^2+y^2+1$, $S: x^2+y^2 \le 1$

$c(t) = (\cos t, \sin t, 0) \to c'(t) = (-\sin t, \cos t, 0)$

$S: \Phi(u,v) = (u\cos v, u\sin v, 0)$

$\vec{T_u} = (\cos v, \sin v, 0)$, $\vec{T_v} = (-u\sin v, u\cos v, 0)$

$\vec{T_u}\times\vec{T_v} = (0,0,u)$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D F[\Phi(u,v)]\cdot(\vec{T_u}\times\vec{T_v})\,du\,dv$$

$$= 0$$

b) Determine the circulation of F around C.

Parameterization: Let $c(t) = (\cos t, \sin t, 0)$

$$\hookrightarrow \int_C F\cdot dS = \int_0^{2\pi} (\cos^2 t, 2\cos t\sin t + \cos t, 0)\cdot(-\sin t, \cos t, 0)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t\sin t + \cos^2 t)\,dt$$

$\int\cos^2 t\sin t\,dt$: Let $u = \cos t \to du = -\sin t\,dt$

$$\hookrightarrow -\int u^2\,du = -\frac{u^3}{3} = -\frac{\cos^3 t}{3}$$

$\int\cos^2 t\,dt$: use reduction formula,

$$\int\cos^n x\,dx = \frac{n-1}{n}\int\cos^{n-2}x\,dx + \frac{\cos^{n-1}x\sin x}{n}$$

$$\hookrightarrow \frac{1}{2}\int\cos^0 t\,dt + \frac{\cos t\sin t}{2}$$

$$= \frac{1}{2}\int dt + \frac{\cos t\sin t}{2}$$

$$\hookrightarrow \int_0^{2\pi}(\cos^2 t\sin t + \cos^2 t)\,dt = \left[-\frac{\cos^3 t}{3} + \frac{t}{2} + \frac{\cos t\sin t}{2}\right]_0^{2\pi}$$

$$= \left[\frac{3[\cos t\sin t + t] - 2\cos^3 t}{6}\right]_0^{2\pi}$$

$$= \pi$$

c) Find the flux of $\nabla\times F$. Verify Stokes' Theorem directly in this case.

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy+x & z \end{vmatrix} = (0, 0, 2y+1)$$

$$\iint_S (\nabla\times F)\cdot dS = \int_0^1\int_0^{2\pi} (0,0,2r\sin\theta+1)\cdot(0,0,1)\,r\,d\theta\,dr$$

$$= \int_0^1\int_0^{2\pi} [2r\sin\theta + 1]\,r\,d\theta\,dr$$

$$= \int_0^1\int_0^{2\pi} [2r^2\sin\theta + r]\,d\theta\,dr$$

$$= \int_0^1 [-2r^2\cos\theta + r\theta]_0^{2\pi}$$

$$= \int_0^1 [(-2r^2 + 2\pi r) - (-2r^2 + 0)]$$

$$= \left[\frac{2\pi r^2}{2}\right]_0^1$$

$$= [\pi r^2]_0^1$$

$$= \pi$$

From the results of b) and c), Stokes' theorem is verified.