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MATH255 T/F

Excellent! 2/2

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HW#9: Sec 8.3 # 2, 4a, 4c, 8, 13, 16, 20, 22, 26, 27a, 29

2.a. $F(x,y) = (\cos xy - xy\sin xy, -x^2\sin xy)$

* check if $F = \nabla f$ $\quad P(x,y) = \cos xy - xy\sin xy \quad Q(x,y) = -x^2\sin xy$

$\frac{\partial P}{\partial y} = -x\sin(xy) - (x\sin(xy) - x^2y\cos(xy)) = -2x\sin(xy) + x^2y\cos(xy)$

$\frac{\partial Q}{\partial x} = -2x\sin xy + x^2y\cos xy$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\therefore$ F is a gradient of a scalar function f

$f(x,y) = \int \cos xy - xy\sin xy \, dx = \int \cos xy \, dx - \int xy\sin xy \, dx$

$\int \cos xy \, dx = \frac{1}{y}\sin(xy)$

$-y\int x\sin xy \, dx \quad u = x \quad v = -\frac{1}{y}\cos(xy) \quad = -y\left[-\frac{1}{y}x\cos(xy) + \int \frac{1}{y}\cos(xy)\,dx\right]$

$du = dx \quad dv = \sin xy \, dx \quad = x\cos(xy) - \int \cos(xy)\,dx$

$= x\cos(xy) - \frac{1}{y}\sin(xy) \quad f(x,y) = \frac{1}{y}\sin(xy) + x\cos(xy) - \frac{1}{y}\sin(xy) = x\cos(xy) + C_1$

$f(x,y) = \int -x^2\sin xy \, dy = -x^2\int \sin xy \, dy = -x^2\left[-\frac{1}{x}\cos xy\right] = x\cos xy + C_2$

$\boxed{f(x,y) = x\cos(xy)}$

2.b. $F(x,y) = (x\sqrt{x^2y^2+1}, y\sqrt{x^2y^2+1})$

$P(x,y) = x\sqrt{x^2y^2+1} \qquad Q(x,y) = y\sqrt{x^2y^2+1}$

$\frac{\partial P}{\partial y} = \frac{1}{2}x(x^2y^2+1)^{-1/2} \cdot 2x^2y = \frac{x^3y}{\sqrt{x^2y^2+1}} \qquad \frac{\partial Q}{\partial x} = \frac{1}{2}y(x^2y^2+1)^{-1/2} \cdot 2xy^2 = \frac{xy^3}{\sqrt{x^2y^2+1}}$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ so $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0$ then F is not a gradient

2.c. $F(x,y) = (2x\cos y + \cos y, -x^2\sin y - x\sin y)$

$\frac{\partial P}{\partial y} = -2x\sin y - \sin y \qquad \frac{\partial Q}{\partial x} = -2x\sin y - \sin y \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\therefore$ F is a gradient field

$f(x,y) = \int 2x\cos y + \cos y \, dx = x^2\cos y + x\cos y + C_1$

$f(x,y) = \int -x^2\sin y - x\sin y \, dy = x^2\cos y + x\cos y + C_2 \quad \boxed{f(x,y) = x^2\cos y + x\cos y}$

4.a. $F(x,y,z) = (e^x\cos y, -e^x\sin y, \pi)$

(i) $\frac{\partial g}{\partial x} = e^x\cos y \qquad \frac{\partial g}{\partial y} = -e^x\sin y \qquad \frac{\partial g}{\partial z} = \pi$

$\int e^x\cos y = e^x\cos y \qquad \int -e^x\sin y = e^x\cos y \qquad \int \pi dz = \pi z \quad \boxed{g(x,y,z) = (e^x\cos y, e^x\cos y, \pi z)}$

(ii) $\nabla \times G = F$ YES there exists a v.f G

$$\nabla \times G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = F = (e^x\cos y, -e^x\sin y, \pi) \qquad \begin{aligned} \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= e^x\cos y \\ \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} &= -e^x\sin y \\ \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= \pi \end{aligned}$$

2/2 Excellent!

4.c $F(x,y,z) = (x^2y^2z^2,\ ye^x,\ xy\cos z)$

$\frac{\partial g}{\partial x} = x^2y^2z^2 \quad \frac{\partial g}{\partial y} = ye^x \quad \frac{\partial g}{\partial z} = xy\cos z$

$\int x^2y^2z^2\,dx = (\tfrac{1}{3})x^3y^2z^2 \quad \int ye^x\,dy = e^x(\tfrac{1}{2})y^2 \quad \int xy\cos z\,dz = xy\sin z$

$\boxed{g(x,y,z) = (\tfrac{1}{3}x^3y^2z^2,\ \tfrac{1}{2}e^xy^2,\ xy\sin z)}$

(ii) $\nabla \times G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = (x^2y^2z^2,\ ye^x,\ xy\cos z)$

$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = x^2y^2z^2$

$\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = ye^x$

$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = xy\cos z$

8. $\int_C F\cdot ds \quad c(t) = (\cos^5 t,\ \sin^3 t,\ t^4) \quad 0 \le t \le \pi \quad F(x,y,z) = (2xyz + \sin x,\ x^2z,\ x^2y)$

$\int_C F\cdot ds = f(c(b)) - f(c(a)) \quad c(\pi) = (-1, 0, \pi^4) \quad c(0) = (1,0,0)$

$\frac{\partial f}{\partial x} = 2xyz + \sin x \quad \frac{\partial f}{\partial y} = x^2z \quad \frac{\partial f}{\partial z} = x^2y$

$\int 2xyz + \sin x\,dx = x^2yz - \cos x + C_1 \quad \int x^2z\,dy = x^2yz + C_2 \quad \int x^2y\,dz = x^2yz + C_3$

$f(x,y,z) = x^2yz - \cos x \quad f(c(t)) = \cos^{10}t\,\sin^3 t\,(t^4) - \cos(\cos^5 t)$

$f(c(\pi)) = \cos^{10}(\pi)\sin^3(\pi)(\pi^4) - \cos(\cos^5(\pi)) = 1\cdot 0\cdot \pi^4 - \cos(-1) = -\cos(1)$

$f(c(0)) = \cos^{10}(0)\sin^3(0)(0^4) - \cos(\cos^5(0)) = -\cos(1) \quad f(c(\pi)) - f(c(0)) = 0$

13. $F(x,y,z) = (e^x\sin y,\ e^x\cos y,\ z^2) \quad c(t) = (\sqrt{t},\ t^3,\ e^{\sqrt{t}}) \quad 0 \le t \le 1$

$\int_C F\cdot ds = f(c(1)) - f(c(0))$

$\frac{\partial f}{\partial x} = e^x\sin y \quad \frac{\partial f}{\partial y} = e^x\cos y \quad \frac{\partial f}{\partial z} = z^2$

$\int e^x\sin y\,dx = e^x\sin y + C_1 \quad \int e^x\cos y\,dy = e^x\sin y + C_2 \quad \int z^2\,dz = \frac{z^3}{3} + C_3$

$f(x,y,z) = e^x\sin y + \frac{z^3}{3} \quad f(c(t)) = e^{\sqrt{t}}\sin(t^3) + \tfrac{1}{3}(e^{\sqrt{t}})^3$

$f(c(1)) = e^{\sqrt{1}}\sin(1) + \tfrac{1}{3}(e^{\sqrt{1}})^3 = e\sin(1) + e^3(\tfrac{1}{3})$

$f(c(0)) = e^0\sin(0) + \tfrac{1}{3}(e^0)^3 = 0 + \tfrac{1}{3} \quad f(c(1)) - f(c(0)) = \boxed{e\sin(1) + \tfrac{1}{3}e^3 - \tfrac{1}{3}}$

16.a. $\int_C \frac{x\,dy - y\,dx}{x^2+y^2} = \int_C \frac{x}{x^2+y^2}\,dy - \frac{y}{x^2+y^2}\,dx$ C is unit circle $\rightarrow x^2+y^2=1$

$x = \cos t \quad y = \sin t$

$x' = -\sin t \quad y' = \cos t$

$= \int_0^{2\pi} \cos t(\cos t) - \sin t(-\sin t)\,dt$

$= \int_0^{2\pi} \cos^2 t + \sin^2 t\,dt = [t]_0^{2\pi} = \boxed{2\pi} ✓$

16.b. $F(x,y) = \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$ is not a conservative field because it's not defined on $(0,0)$

16.c. $P(x,y) = \frac{-y}{x^2+y^2} \quad Q(x,y) = \frac{x}{x^2+y^2} \quad \frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ it doesn't contradict the corollary because $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

but the corollary can be false if it's not defined on at least one point

20. $G = (G_1, G_2, G_3)$ $\quad G_1 = \int_0^x F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt$

$G_2 = -\int_0^x F_1(x,y,t)\,dt \quad G_3 = 0$

$$\text{curl } G = \nabla \times G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = \left(\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2,\ \frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3,\ \frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1\right)$$

$\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2 = \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\int_0^x F_1(x,y,t)\,dt = F_1(x,y,z)$

$\frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3 = \frac{\partial}{\partial z}\left(\int_0^x F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt\right) - \frac{\partial}{\partial x}(0) = F_2(x,y,z)$

$\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1 = \frac{\partial}{\partial x}\left(-\int_0^x F_1(x,y,t)\,dt\right) - \frac{\partial}{\partial y}\left(\int_0^x F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt\right)$

$= -\frac{\partial}{\partial x}\int_0^x F_1(x,y,t)\,dt - \frac{\partial}{\partial y}\int_0^x F_2(x,y,t)\,dt + F_3(x,y,0)$

$\text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \frac{\partial F_3}{\partial z} = -\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}$

$\nabla \times G = \left(F_1(x,y,z), F_2(x,y,z), \frac{\partial}{\partial z}\int_0^z F_3(x,y,t)\,dt + F_3(x,y,0)\right)$

$= (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) - F_3(x,y,0) + F_3(x,y,0)) = F$ ✓

22. $F = (xz, -yz, y)$

$\nabla \cdot F = z - z + 0 = 0$ ✓

$$\nabla \times G = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = (xz, -yz, y)$$

$\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2 = xz$

$\frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3 = -yz$

$\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1 = y$

let $G_3 = xyz \Rightarrow \frac{\partial}{\partial y}(xyz) = xz \qquad xz - \frac{\partial}{\partial z}G_2 = xz$

$\Rightarrow \frac{\partial}{\partial x}(xyz) = yz \qquad \frac{\partial}{\partial z}G_1 - yz = -yz$

let $G_2 = xy \Rightarrow \frac{\partial}{\partial z}(xy) = 0 \qquad \frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2 = xz \qquad$ let $G_1 = x \Rightarrow \frac{\partial}{\partial z}G_1 = 0$

$\Rightarrow \frac{\partial}{\partial x}(xy) = y \qquad y - \frac{\partial}{\partial y}G_1 = y \qquad \Rightarrow \frac{\partial}{\partial y}G_1 = 0$

$\boxed{G = (x, xy, xyz)}$

26. (i) implies (ii)

$F = \nabla f \quad F_1 = \frac{\partial f}{\partial x} \quad F_2 = \frac{\partial f}{\partial y} = f(x,y,z) = F_2(y) + C_2 \qquad f(x,y,z) = F_1(x) + F_2(y) + C$

$f(x,y,z) = F_1(x) + C_1$

$\int_{C_1} F \cdot ds = (f(x,\cancel{y,0}) - f(0,0,0) + f(x,y,z) - \cancel{f(x,y,0)}) = f(x,y,z) - f(0,0,0)$

$= F_1(x) + F_2(y) + c - (F_1(0) + F_2(0) + C) = F_1(x) + F_2(y)$ from $(0,0,0) \to (x,y,0) \to (x,y,z)$

$(0,0,0) \to (x,y,z)$

$\int_{C_2} F \cdot ds = f(x,y,z) - f(0,0,0) = F_1(x) + F_2(y) + C - (F_1(0) + F_2(0) + C)$

$= F_1(x) + F_2(y)$

Therefore both paths give the same integral

27. a. $F(x,y,z) = (-y, x, 0)$

Irrotational: curl $F = 0$ $\quad \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0-0, 0-0, 1+1) = (0,0,2) \neq 0$

$\therefore$ F is rotational

27. b. $C(t)$ = path of the cork

$F(c(t)) = c'(t)$ let $c(t) = (x(t), y(t), z(t))$

$F(c(t)) = F(x(t), y(t), z(t)) = (-y, x, 0) = c'(t) \quad x'(t) = -y \quad y'(t) = x \quad z'(t) = 0$

Because z is a constant it shows the cork is moving // to the xy plane

$x'' + x = 0 \quad y'' + y = 0$ then $x = A\cos t + B\sin t \quad y = C\cos t + D\sin t$

$x'(t) = -A\sin t + B\cos t = -C\cos t - D\sin t = -y \Rightarrow D = A \quad -C = B$

$x^2 + y^2 = (A\cos t + B\sin t)^2 + (-B\cos t + A\sin t)^2 = A^2\cos^2 t + \cancel{2AB\sin t\cos t} + B^2\sin^2 t + B^2\cos^2 t$

$- 2A\cancel{B\sin t}\cos t + A^2\sin^2 t = A^2 + B^2 \quad \therefore$ it's a circle

27. c. clockwise because of the parametrized values

29. a. $F = \frac{-GmMr}{r^3}$

let $r^2 = x^2+y^2+z^2 \Rightarrow F = \frac{-GmM(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$

$\text{div } F = \nabla \cdot F = -GmM\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3} = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial F_2}{\partial y} = \frac{(x^2+y^2+z^2)^{3/2}(1) - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial F_3}{\partial z} = \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}} \qquad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = \boxed{0} \checkmark$$

29. b. if $F = \nabla \times G$ then $\iint_S F \cdot ds = 0$

$\iint_S F \cdot dS = -GMm \iint_S \frac{(x,y,z)}{\|x^2+y^2+z^2\|^3} \cdot n\,ds = -4\pi GMm$ because $\|r\| = 1$ and $r = n$

$\therefore \iint_S F \cdot ds \neq 0 \Rightarrow F \neq \nabla \times G$