Aunt Debi
Math 255
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Homework #9
Section 8.3

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2) a) $F(x,y)=(\cos xy - xy\sin xy, -x^2\sin xy)$

$\int(\cos xy - xy\sin xy)dx \rightarrow \boxed{f = x\cos(yx)+C}$

$\frac{\partial}{\partial x}(f) = \cos xy - xy\sin xy$, $\frac{\partial}{\partial y}(f) = x^2\sin xy$ ✓

$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \cos(xy) - xy\sin(xy) & -x^2\sin xy & 0 \end{vmatrix} = (0,0,0) \rightarrow$ conservative

b) $F(x,y) = (x\sqrt{x^2y^2+1}, y\sqrt{x^2y^2+1})$

$\nabla\times F = \begin{vmatrix} i & j \\ dx & dy \\ x\sqrt{x^2y^2+1} & y\sqrt{x^2y^2+1} \end{vmatrix} = \frac{xy^3 - x^3y}{\sqrt{x^2y^2+1}} \neq 0$

∴ Not conservative
and F is not a gradient field.

c) $F(x,y) = (2x\cos y + \cos y, -(x^2\sin y + x\sin y))$

$\nabla\times F = \begin{vmatrix} i & j \\ \partial x & \partial y \\ 2x\cos y + \cos y & -(x^2\sin y + x\sin y) \end{vmatrix} = (0) \rightarrow$ conservative

$\int 2x\cos y + \cos y)dx = \cos(y)x(x+1) + C$

$\int -(x^2\sin y + x\sin y)dy = \cos(y)x(x+1) + C$

$\boxed{f = (x^2+x)\cos y + C}$

4) a) $F(x,y,z) = (e^x\cos y, -e^x\sin y, \pi)$

$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ e^x\cos y & -e^x\sin y & \pi \end{vmatrix} = (0,0,0) \rightarrow$ conservative $\rightarrow$ since $\nabla\times F = 0$, F is conservative and g exists

curl G = F?

$\nabla\times G = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ G_1 & G_2 & G_3 \end{vmatrix} = (\underbrace{\tfrac{\partial}{\partial y}G_3 - \tfrac{\partial}{\partial z}G_2}_{e^x\cos y}, \underbrace{\tfrac{\partial}{\partial z}G_1 - \tfrac{\partial}{\partial x}G_3}_{-e^x\sin y}, \underbrace{\tfrac{\partial}{\partial x}G_2 - \tfrac{\partial}{\partial y}G_1}_{\pi})$

$G_3 = e^x\sin y$, $G_2 = \pi x$, $G_1 = x$

$\nabla\times G = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x & \pi x & e^x\sin y \end{vmatrix} = (e^x\cos y, -e^x\sin y, \pi) = F$

∴ There does exist a vector field G s.t. $\nabla\times G = F$

c) $F(x,y,z)=(x^2y^2z^2, ye^x, xy\cos z)$

$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ x^2y^2z^2 & ye^x & xy\cos z \end{vmatrix} \neq (0,0,0) \rightarrow$ Not conservative

There does not exist a function g s.t. $\nabla g = F$.

$\nabla \times G = F$?

$\nabla \times G = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ G_1 & G_2 & G_3 \end{vmatrix} = \left(\underbrace{\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2}_{x^2y^2z^2}, \underbrace{\frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3}_{ye^x}, \underbrace{\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1}_{xy\cos z}\right)$

$G_2 = -x^2y^2\frac{z^3}{3}$, $G_3 = $ no y

$G_3 = -ye^x$, $G_1 = ye^x z$

There does not exist a vector field G s.t. $\nabla \times G = F$.

8) $c(t) = (\cos^5 t, \sin^3 t, t^4)$, $t \in [0, \pi]$, $F(x,y,z) = (2xyz + \sin x, x^2z, x^2y)$

$\int_c F \cdot ds = \int_c F_1\,dx + F_2\,dy + F_3\,dz$

$= \int (2xyz + \sin x)\,dx + (x^2z)\,dy + (x^2y)\,dz$

$x^2yz - \cos x + x^2yz + x^2yz \Big|_{t=0}^{t=\pi}$

$3x^2yz - \cos x \Big|_{t=0}^{t=\pi}$

$3(\cos^{10} t)(\sin^3 t)(t^4) - \cos(\cos^5 t)\Big|_0^\pi$

$3(-1)^{10}(0)(\pi^4) - \cos(-1)^5 - [3(1)^{10}(0)(0^4) - \cos(1)^5]$

$0 - \cos(-1) + \cos(1) = \boxed{-\cos(-1) + \cos(1)}$

13) $F(x,y,z) = (e^x \sin y, e^x \cos y, z^2)$. $c(t) = (\sqrt{t}, t^3, e^{\sqrt{t}})$, $t \in [0,1]$

$\int_c F \cdot ds = \int_c F_1\,dx + F_2\,dy + F_3\,dz$

$= \int e^x \sin y\,dx + e^x \cos y\,dy + z^2\,dz$

$= \sin y e^x + e^x \sin y + \frac{z^3}{3}$

$= \sin y e^x + \frac{z^3}{3}\Big|_{t=0}^{t=1}$

$= \sin(t^3) e^{\sqrt{t}} + \frac{(e^{\sqrt{t}})^3}{3}\Big|_0^1$

$= \left(e\sin(1) + \frac{e^3}{3}\right) - \left(\frac{1}{3}\right)$

$= \boxed{e\sin(1) - \frac{e^3}{3} - \frac{1}{3}}$

16a) Show that $\int_C (x\,dy - y\,dx)/(x^2+y^2) = 2\pi$, C: unit circle

$$\int_C \frac{x\,dy - y\,dx}{x^2+y^2}$$

$x = \cos t,\ y = \sin t$

$dx = -\sin t,\ dy = \cos t,\quad t \in [0, 2\pi]$

$$\int \frac{\cos t}{\cos^2 t + \sin^2 t}\,dy - \frac{\sin t}{\cos^2 t + \sin^2 t}\,dx$$

$$\int \cos t \cos t + \sin t \sin t\,dt$$

$$\int 1\,dt = t\Big|_0^{2\pi} = \boxed{2\pi}$$

b) Conclude that $\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is not a conservative field.

From 16a, $\int_C \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy = 2\pi$

$$\int_C F_1\,dx + F_2\,dx = 2\pi$$

$$\int_C F\cdot ds = 2\pi$$

A conservative field has $\int_C F\cdot ds = 0 \neq 2\pi$

$\therefore$ Since $2\pi \neq 0$, the vector field $\left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is not a conservative field.

c) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Does this contradict the corollary to theorem 7?

$P = \frac{-y}{x^2+y^2}$ $\qquad\qquad$ $Q = \frac{x}{x^2+y^2}$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$\frac{y^2-x^2}{(y^2+x^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{y^2-x^2}{(y^2+x^2)^2} = \frac{y^2-x^2}{(y^2+x^2)^2}$$

It does not contradict the corollary b/c there exists an f s.t. $\nabla f = F$

$f = \arctan\left(\frac{y}{x}\right)$

$\nabla f = \left(\frac{-y}{x^2+y^2}, \frac{x}{(x^2+y^2)}\right) = F$

But F is not C' at all points such as $(0,0)$. Therefore the corollary does not apply.

20) Prove Theorem 8: If F is a C^1 vector field on all of $\mathbb{R}^3$ w/ $\operatorname{div} F = 0$ then there exists a C^1 vector field G w/ $F = \operatorname{curl} G$.

Define $G = (G_1, G_2\ G_3)$ by

$$G_1(x,y,z) = \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt$$

$$G_2(x,y,z) = -\int_0^z F_1(x,y,t)\,dt$$

$$G_3(x,y,z) = 0$$

$$\nabla \times G = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt & -\int_0^z F_2(x,y,t)\,dt & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial z}\int_0^z F_1(x,y,t)\,dt,\ \frac{\partial}{\partial z}\left[\int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt\right],\right.$$
$$\left.\left(\frac{\partial}{\partial x}\left(-\int_0^z F_1(x,y,t)\,dt\right) - \frac{\partial}{\partial y}\left(\int_0^z F_2(x,y,t)\,dt - \int_0^y F_3(x,t,0)\,dt\right)\right)\right)$$

k^{th} component $= \left(\frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1\right) = -\int_0^z \frac{\partial}{\partial x}(F_1(x,y,t)\,dt) - \int_0^z \frac{\partial}{\partial y}(F_2(x,y,t)\,dt - F_3(x,y,0))$

$$= \int_0^z\left(-\frac{\partial}{\partial x}(F_1(x,y,t)) - \frac{\partial}{\partial y}(F_2(x,y,t)\right)dt + F_3(x,y,0)$$

$$\operatorname{Div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$\frac{\partial F_3}{\partial z} = -\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}$$

Substitute $\frac{\partial F_3}{\partial z}$ for $-\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}$:

$$= \int_0^z\left(\frac{\partial F_3(x,y,t)}{\partial z}\right)dt + F_3(x,y,0)$$

$$= F_3(x,y,t)\Big|_0^z + F_3(z,y,0)$$

$$= [F_3(x,y,z) - F_3(x,y,0)] + F_3(x,y,0)$$

k^{th} component $= F_3(x,y,z)$

i^{th} component $= \int_0^z \frac{\partial}{\partial z}(F(x,y,t))\,dt = F_1(x,y,z)$

j^{th} component $= \int_0^z \frac{\partial}{\partial z}(F_2(x,y,t) - F_3(x,t,0)\,dt)\,dt = F_2(x,y,z)$

$\therefore \nabla \times G = F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) = F$

22) Let $F = (xz, -yz, y)$. Verify that $\nabla \cdot G = 0$. Find a G s.t. $F = \nabla \times G$.

$\nabla \cdot F = z - z = 0$

$$\nabla \times G = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ G_1 & G_2 & G_3 \end{vmatrix} = \left(\frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2,\ \frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3,\ \frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1\right)$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}G_3 - \frac{\partial}{\partial z}G_2 &= xz \\ \frac{\partial}{\partial z}G_1 - \frac{\partial}{\partial x}G_3 &= -yz \\ \frac{\partial}{\partial x}G_2 - \frac{\partial}{\partial y}G_1 &= y \end{aligned}\right\} \boxed{G = (x, xy, xyz)}$$

26) By using different $(0,0,0)$ to (x,y,z), show that the function f defined in the proof of thm. 7 for "condition (ii) implies condition (iii)" satisfies $\frac{\partial f}{\partial x} = F_1$ and $\frac{\partial f}{\partial y} = F_2$

Let C be the path from $(0,0,0)$ to (t,y,z) to $(x,y,0)$ to (x,y,z) so that

$$f(x,y,z) = \int_0^x F_1(t,y,z)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt.$$

From the Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = F_1(x,y,z)$$

let C be the path from $(0,0,0)$ to $(x,0,0)$ to (x,t,z) to (x,y,z)

so that

$$f(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,z)\,dt + \int_0^z F_3(x,y,t)\,dt$$

From the Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial y} = F_2(x,y,z)$$

27) a) Let F be a vector field on $\mathbb{R}^3$ given by $F = (-y, x)$. Show that F is rotational.

A vector field F w/ curl $F = 0$ is irrotational.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ -y & x & 0 \end{vmatrix} = (0,0,2) \neq 0$$

$\therefore F$ is not irrotational.

29) Let $F = -\frac{GmMr}{r^3}$ be the gravitational force field defined on $\mathbb{R}^3 \setminus \{0\}$

a) Show that $\operatorname{div} F = 0$

$$F = \frac{-GmMr}{\|r\|^3} \quad \text{where } r = (x,y,z)$$

$$F = \frac{-GmM(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -GmM\left[\frac{x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^{5/2}}\right] = 0$$

$$= -GmM(0) = \boxed{0}$$

b) Show that $F \neq \text{curl } G$ for any C^1 vector field G on $\mathbb{R}^3 \setminus \{0\}$

Let S be an oriented surface w/ oriented boundary ∂S.

$$\iint_S \text{curl } G \cdot n \, dA = \int_{\partial S} G \cdot ds$$

However, if S is boundary-less then $\iint_S \text{curl } G \cdot n \, dA = 0$.

To prove that F is not the curl of another vector field G, we need to find a boundary less surface S s.t. $\iint_S F \cdot n \, dA \neq 0$

Let S^2 be the unit sphere and be boundary less.

$$F = G_m M\left(\frac{-r}{\|r\|^3}\right)$$

Then
$$\begin{aligned}\iint_{S^2} F \cdot n \, dA &= -G_m M \iint_{S^2} \frac{r \cdot r}{\|r\|^3} dA \quad \text{b/c } n = r \\ &= -G_m M \iint_{S^2} \frac{1}{\|r\|} dA \\ &= -G_m M \iint_{S^2} 1 \, dA \\ &= -4\pi G_m M \rightarrow \text{area of } S^2 \text{ is } 4\pi\end{aligned}$$

Since $-4\pi G_m M \neq 0$, $F \neq \text{curl } G$