TÉCNICO LISBOA

#1. $\nabla \times (f\vec{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \left( \frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2),\ \frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3),\ \frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1) \right)$

$= \left( \frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2 + f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right),\ \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3 + f\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right),\ \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1 + f\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \right)$

$= \nabla f \times \vec{F} + f\,\nabla \times \vec{F}.$ ✓

#2. $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \partial_x(x) + \partial_y(y) + \partial_z(z^2) = 1+1+2z = 2+2z.$

$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z^2 \end{vmatrix} = 0.$

#3. $\int_\alpha \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\alpha(t)) \cdot \alpha'(t)\, dt = \int_0^{2\pi} (\cos t, \sin t, t^2) \cdot (-\sin t, \cos t, 1)\, dt$

$= \int_0^{2\pi} t^2\, dt = \frac{1}{3}t^3 \Big|_0^{2\pi} = \frac{8}{3}\pi^3.$

#4. $S$ is regular at $\Phi(u,v)$ $\Leftrightarrow$ $\vec{T}_u \times \vec{T}_v \neq \vec{0}$ at $\Phi(u,v) \in S$.

$\vec{T}_u = (-\sin u, \cos u, 0)$
$\vec{T}_v = (0, 0, 1)$

$\vec{T}_u \times \vec{T}_v = (\cos u, \sin u, 0) \neq 0$ for any $(u,v) \in [0,2\pi] \times [0,1]$.

Hence $S$ is regular at $\Phi(u,v)$ for all $(u,v) \in D$.

#5. $A(S) = \iint_S dS = \iint_D \|\vec{T}_u \times \vec{T}_v\|\, du\, dv = \int_0^{2\pi}\int_0^1 1 \cdot dv\, du = 2\pi.$

#6. $\iint_S z^2\, dS = \iint_D \left(z(\Phi(u,v))\right)^2 \|\vec{T}_u \times \vec{T}_v\|\, du\, dv = \int_0^{2\pi}\int_0^1 v^2\, dv\, du = \frac{2\pi}{3}.$

#7. $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot \vec{T}_u \times \vec{T}_v \, du dv = \iint_D (\cos u, \sin u, v^2)\cdot(\cos u, \sin u, 0)\, du dv$

$$= \int_0^{2\pi}\int_0^1 1 \, dv\, du = 2\pi.$$

#8.

$E = \vec{T}_u \cdot \vec{T}_u = 1$

$F = \vec{T}_u \cdot \vec{T}_v = 0$

$G = \vec{T}_v \cdot \vec{T}_v = 1$

$\vec{N} = \dfrac{\vec{T}_u \times \vec{T}_v}{\|\vec{T}_u \times \vec{T}_v\|} = (\cos u, \sin u, 0)$

$L = \vec{N}\cdot \vec{T}_{uu} = -1$

$M = \vec{N}\cdot \vec{T}_{uv} = 0$

$N = \vec{N}\cdot \vec{T}_{vv} = 0$

$\vec{T}_{uu} = (-\cos u, -\sin u, 0)$

$\vec{T}_{uv} = (0, 0, 0)$

$\vec{T}_{vv} = (0, 0, 0)$

At an arbitrary point $P = \Phi(u,v)$,

$$K(P) = \frac{LN - M^2}{EG - F^2} = 0 \qquad H(P) = \frac{GL + EN - 2FM}{2(EG - F^2)} = -\frac{1}{2}.$$

#9. Heat flux $\vec{F} = -\nabla T = (-6x, 0, -6z)$

Let $\Phi : (u,v) \mapsto (\sqrt{2}\cos u, v, \sqrt{2}\sin u)$ be a parametrization (we choose) of the given surface, defined on $[0,2\pi]\times[0,2]$.

The heat flux across the surface $\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\Phi(u,v))\cdot \vec{T}_u \times \vec{T}_v \, du dv = (*)$

Here $\vec{T}_u = (-\sqrt{2}\sin u, 0, \sqrt{2}\cos u)$

$\vec{T}_v = (0, 1, 0)$ $\qquad \vec{T}_u \times \vec{T}_v = (-\sqrt{2}\cos u, 0, -\sqrt{2}\sin u)$

$\vec{F}(\Phi(u,v)) = (-6\sqrt{2}\cos u, 0, -6\sqrt{2}\sin u)$

$$(*) = \int_0^{2\pi}\int_0^2 12 \, dv\, du = 12 \cdot 2 \cdot 2\pi = 48\pi.$$