Final Exam Review Sheet (May 12th Updated Version)
MATH 25500 Section 01
Exam date and time: 19th May 2017, 14:10–15:25

Instructions: This exam is closed-book and closed-notes. Electronic devices are also not permitted to use. Talking to other students will not be allowed. Answers without justifications and/or calculation steps may receive no score.

1. Evaluate the integral

$$\int_C (2x^3 - y^3)dx + (x^3 + y^3)dy$$

where C is the unit circle in $\mathbb{R}^2$.

2. Use Green's theorem to calculate the area of the circle with radius R centered at $(0, 0)$ in $\mathbb{R}^2$.

3. Compute the line integral

$$\int_C x^3 dx - y^3 dy$$

along the unit circle C in $\mathbb{R}^2$.

4. If C is a closed curve that is the boundary of a surface S, and f and g are C^2 functions, show that

$$\int_C f\nabla g \cdot d\overrightarrow{r} = \iint_S (\nabla f \times \nabla g) \cdot d\overrightarrow{S}.$$

5. Prove or disprove the following statement: If a C^1-vector field $\overrightarrow{F}$ is defined on any convex region D in $\mathbb{R}^2$ satisfying that $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for any loop C in D, then there exists a function f defined on D satisfying that $\overrightarrow{F} = \nabla f$.

6. Evaluate $\iint_S \overrightarrow{F} \cdot d\overrightarrow{S}$, where $\overrightarrow{F} = x^3\overrightarrow{i} + y^3\overrightarrow{j} + z^3\overrightarrow{k}$ and S is the surface of the unit sphere in $\mathbb{R}^3$.

7. Calculate $\iint_S \frac{\overrightarrow{r}}{r^3} \cdot d\overrightarrow{S}$ in the case that S is the boundary of a subset V in $\mathbb{R}^3$ that the Gauss' divergence theorem applies, and $(0, 0, 0)$ is contained in $\mathbb{R}^3$. Here $\overrightarrow{r} := (x, y, z)$.

8. Let $\omega = yzdx + zxdy + xydz$. Compute $d\omega$.

9. (1) Let $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map. Assume that $m < n$ and let ω be a k-form in $\mathbb{R}^n$, with $k > m$. Show that $f^*\omega = 0$.

(2) Let ω be a differential k-form with k an odd positive integer. Prove that $\omega \wedge \omega = 0$.

10. Given a k-form ω in $\mathbb{R}^n$ we will define an $(n-k)$-form $*\omega$ by setting

$$*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma (dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, $(i_1, \cdots, i_k, j_1, \cdots, j_{n-k})$ is a permutation of $(1, 2, \cdots, n)$ and σ is 0 or 1 according to the permutation is even or odd, respectively. Show that:

(1) If $\omega = a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{23} dx_2 \wedge dx_3$ is a 2-form in $\mathbb{R}^3$, then

$$*\omega = a_{12} dx_3 - a_{13} dx_2 + a_{23} dx_1.$$

(2) If $\omega = a_1 dx_1 + a_2 dx_2$ is an 1-form in $\mathbb{R}^2$, then

$$*\omega = a_1 dx_2 - a_2 dx_1.$$

(3) $** \omega = (-1)^{k(n-k)} \omega$.

Note: Let an array of n numbers $(1 \quad 2 \cdots n)$ be given. An array of numbers consisting of $1, 2, \cdots, n$ in arbitrary order is called an **even permutation** (resp. **odd permutation**) if it is obtained by even (resp. odd) number of swapping two adjacent numbers. The array $(1 \quad 2 \cdots n)$ itself is an even permutation. For example, $(1 \quad 3 \quad 2)$ is an odd permutation and $(3 \quad 2 \quad 1)$ is an even permutation. Also $(2 \quad 1)$ is an odd permutation and $(1 \quad 2)$ is an even permutation.