2025. 8. 5.

Lecture 1. Conics, invariants, center

1. Standard conics

(1) parabola 포물선

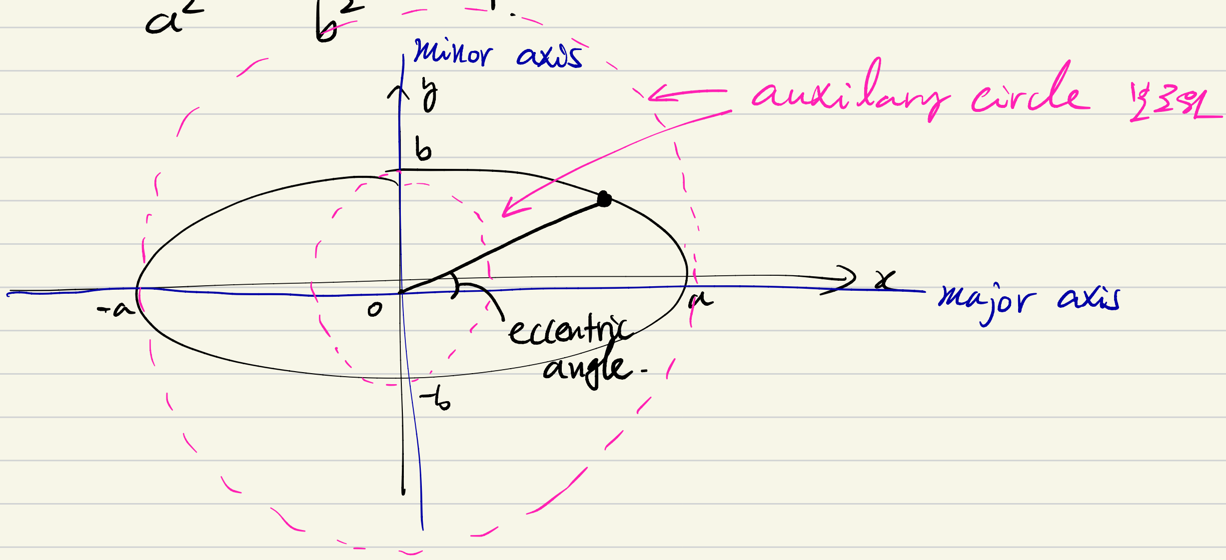
Def: Let $a>0$. The Standard parabola with modulus a is

$$y^2 = 4ax.$$

(2) Ellipse 타원

Def: Let $a>b>0$. The Standard real ellipse with moduli a, b is

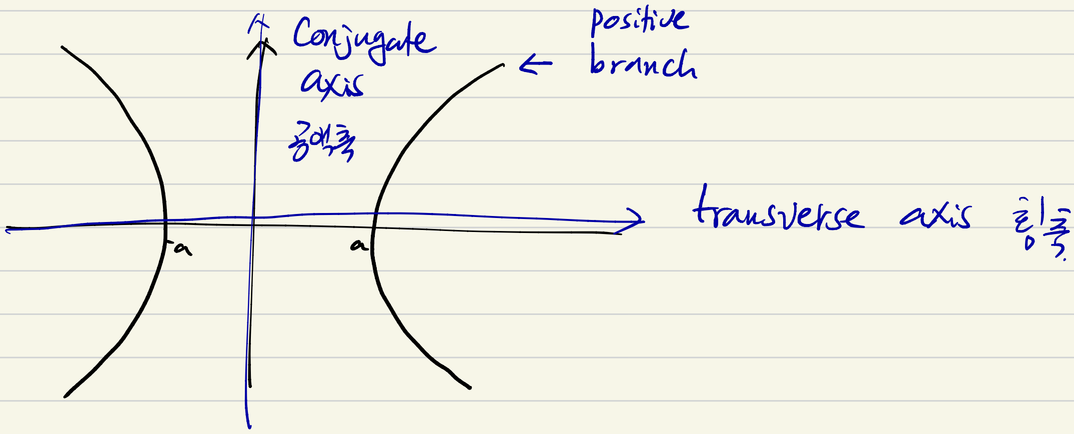
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$



Note In our definition, circles are not ellipses, but circles are obtained by taking limits $b \to a$

(3) Hyperbola 쌍곡선

Def: Let $a, b > 0$. The <u>standard hyperbola</u> with moduli a, b is the conic
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



2. Matrices and invariants

Def: A <u>quadratic function</u> $Q : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$(x, y) \rightsquigarrow Q(x, y)$

such that $Q(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.

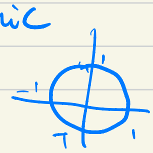
E.g. The unit circle $x^2+y^2=1$ can be viewed as the set
$$Z(Q) = \{ (x,y) \in \mathbb{R}\times\mathbb{R} : Q(x,y) = x^2+y^2-1 = 0 \}$$

Two quadratic functions Q_1, Q_2 are <u>scalar multiples</u> (실수배) of each other if $Q_1 = \lambda Q_2$ for some $\lambda \in \mathbb{R} \setminus \{0\}$

Def: A <u>Conic</u> is a set of quadratic functions each of which is a scalar multiple of the other.

$[\, Q_1 = x^2+y^2-1 \rightarrow Q_{2000} = 2000x^2+2000y^2-2000 \rightarrow Q_{2025} = 2025x^2+2025y^2-2025 \rightarrow Q_{-3} = -3x^2-3y^2+3 \cdots \,]$ Conic

quadratic function



Def Let $Q(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$ be a conic.

tau

The trace invariant τ is $a+b$

The delta invariant δ is $ab-h^2$

The discriminant Δ is $a(bc-f^2)+h(fg-hc)+g(hf-gb)$

Note Matrix interpretation.

Need to know what determinants are.
행렬식

What is matrix? A rectangular arrangement of numbers

$$\begin{array}{cl} \begin{pmatrix} 2 & 3 & 7 \\ 1 & 5 & 0 \end{pmatrix} & \begin{array}{l} \to 1\text{행} \\ \to 2\text{행} \end{array} \\ \begin{array}{ccc} 1\text{열} & 2\text{열} & 3\text{열} \end{array} & \end{array} \qquad \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$$

2개행 × 3개열

2 × 3 행렬

3 × 3 행렬

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

5 × 1 행렬

행렬의 연산

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a \pm x & b \pm y \\ c \pm z & d \pm w \end{pmatrix}$$

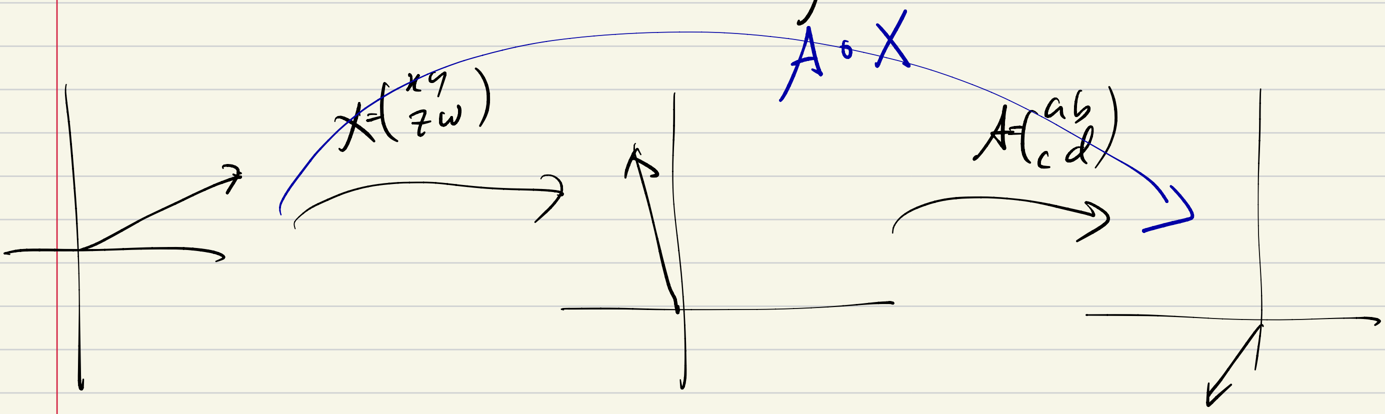
행렬의 곱

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax & by \\ cz & dw \end{pmatrix}$$

Useless!

$$\| $$

$$\begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$$



Homework: Convince Yourself : Matrix addition is Commutative (교환법칙 성립) and associative (결합법칙 성립)

Also, matrix multiplication is associative. However, matrix multiplication is not Commutative. Find an example.

Matrix inverse? For example in $\mathbb{R}$, knowing $x \div y$ means knowing y satisfying $x \cdot y = 1$

Let A be an $n \times n$ matrix its inverse A^{-1} is a matrix satisfying

$$AA^{-1} = I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix}$$

identity matrix 항등행렬

Exercise Find $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Answer $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Def If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = ad - bc$

or $\det(A)$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} |A| = &+a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &-a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &+a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$n \times n$ 행렬

2×2 행렬식 연산 필요수 $\propto n!$

- **Matrix interpretation of invariants.**

Let $Q(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + e$

$\downarrow$ Associated matrix

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$: Symmetric matrix.

$z = (x \ \ y \ \ 1)$

$$Q(x,y) \overset{\text{You check!}}{=} \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$1 \times 3 \qquad 3 \times 3 \qquad 3 \times 1$

Let's look at $\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

Aside: $A = (a_{ij})$, $1 \le i, j \le n$

$a_{11} + a_{22} + a_{33} + a_{44} + \dots + a_{nn} =: \operatorname{trace}(A)$

$$\tau = \operatorname{trace}\begin{pmatrix} a & h \\ h & b \end{pmatrix} \qquad \delta = \det\begin{pmatrix} a & h \\ h & b \end{pmatrix} \qquad \Delta = \det A$$

<u>Note</u>: The invariants τ, δ, Δ are not invariant under scalar multiplication. However, the following important equalities and inequalities remain invariant:

$\tau = 0$	$\delta = 0$	$\Delta \neq 0$
$\tau \neq 0$	$\delta > 0$	$\Delta = 0$
	$\delta < 0$	

$$\left(\because \begin{array}{l} Q \rightsquigarrow \lambda Q \\ \tau \rightsquigarrow \lambda \tau \\ \delta \rightsquigarrow \lambda^2 \delta \\ \Delta \longrightarrow \lambda^3 \Delta \end{array} \right)$$

Examples (1) $Q(x,y) = x^2+y^2 \overset{+1}{\underset{+0}{-1}}$

Associated matrix $\begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & -1 \end{pmatrix}$

$\tau = 2$, $\delta = 1$, $\Delta = -1$

$\tau = 2$, $\delta = 1$, $\Delta = 0$

$\tau = 2$, $\delta = 1$, $\Delta = 1$

(2) $Q(x,y) = y^2 - 4ax$ $a>0$

Associated matrix $\begin{pmatrix} 0 & 0 & -2a \\ 0 & 1 & 0 \\ -2a & 0 & 0 \end{pmatrix}$

$\tau = 1$

$\delta = 0$

$\Delta = -4a^2 \neq 0$.

(3) $Q(x,y) = \frac{x^2}{a^2} \pm \frac{y^2}{b^2} - 1$ ellipse $\delta > 0$, hyperbola $\delta < 0$

Associated matrix $\begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\tau = \frac{1}{a^2} \pm \frac{1}{b^2}$

$\delta = \frac{\pm 1}{a^2 b^2} > 0$ < 0.

$\Delta \neq 0$.

$\Delta \neq 0$		Standard $\overset{\text{Cost!!}}{\Longrightarrow}$ "Invariance theorem"	general
	$\delta > 0$	ellipse	ellipse
	$\delta = 0$	parabola	parabola
	$\delta < 0$	hyperbola	hyperbola

Def: A conic Q is said to be degenerate(퇴화) if $\Delta = 0$

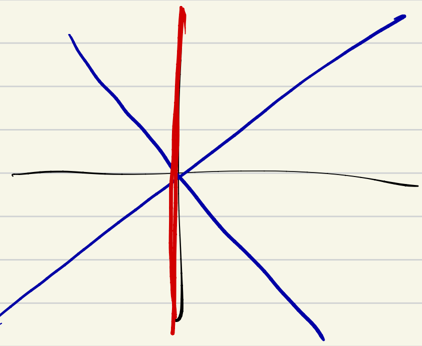
Examples (of degenerate conics)

$Q(x,y) = 0$

$Q(x,y) = x^2 - y^2$

$Q(x,y) = x^2$

$Q(x,y) = x^2 + y^2$: irreducible.

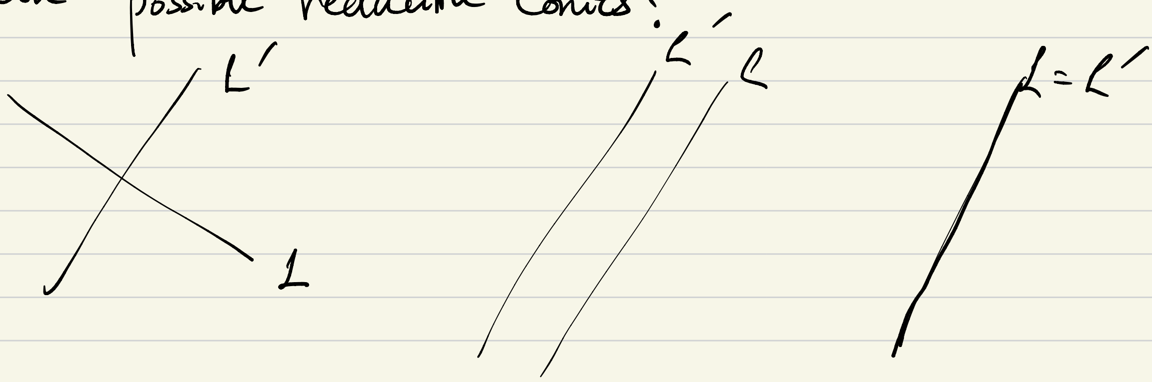


3. Reducible conics

A line L: $ax+by+c$

Def: A conic Q is reducible if there exists L, L': lines such that $Q = LL'$ $(= (ax+by+c)(a'x+b'y+c'))$.
Here L, L' are called components of Q and $Q=0$ is the joint equation of L, L'.
Otherwise, Q is irreducible.

What are possible reducible conics?



Exercise Is $Q(x,y) = \underline{x^2 - xy - 2y^2} + 2x + 5y - 3$ a reducible conic?

$$Q(x,y) = (1x - 1y + \underbrace{\alpha}_{-1})(1x - 2y + \underbrace{\beta}_{+3})$$

보조정리

$\beta \neq 0$

Lemma: Let Q be a conic and $\mathcal{L}$ a line not parallel to the y-axis. Then there is a unique line $\mathcal{L}'$ and a unique quadratic function $J(x)$ s.t.

$$Q(x,y) = \mathcal{L}(x,y)\,\mathcal{L}'(x,y) + J(x)$$

정리

Theorem (Component lemma) Suppose every point on the line $\mathcal{L}$ lies on Q: a conic. Then Q must be a reducible conic.

i.e. there exists $\mathcal{L}'$ a line such that

$$Q = \mathcal{L}\mathcal{L}'$$

Proof: Assume $\mathcal{L}$: not parallel to y-axis.

$$\cancel{Q}^{(x,y)} = \cancel{\mathcal{L}}^{(x,y)}\mathcal{L}' + J(x) \quad \forall x \qquad \text{for some } \mathcal{L}',\ J(x)$$

In the assumption, For every x, there is a unique y s.t. $\mathcal{L}(x,y) = 0 \Rightarrow Q(x,y) = 0$

Therefore $J(x) \equiv 0$ for every $x \in \mathbb{R}$

$$Q = \mathcal{L}\mathcal{L}' \qquad \blacksquare$$

Sketch of proof of Lemma:

$\mathcal{L} = \alpha x + \beta y + \gamma$

$\mathcal{L}' = \alpha' x + \beta' y + \gamma'$

$\mathcal{L}\mathcal{L}'$ 전개

y^2, xy, y의 계수를 같도록

$\beta\beta'$, $\alpha\beta' + \alpha'\beta$, $\beta\gamma' + \beta'\gamma$

Q(x,y)의 해당 계수와 비교

$$\begin{cases} \beta\beta' = b \\ \beta\alpha' + \alpha\beta' = 2h \\ \gamma\beta' + \beta\gamma' = 2f. \end{cases}$$

$$-\beta^3 = \det\begin{pmatrix} 0 & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & \gamma & \beta \end{pmatrix} \neq 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

has a unique sol

$\Leftrightarrow \det(A) \neq 0$

$$\therefore J(x) := Q(x,0) - L(x,0)L'(x,0).$$