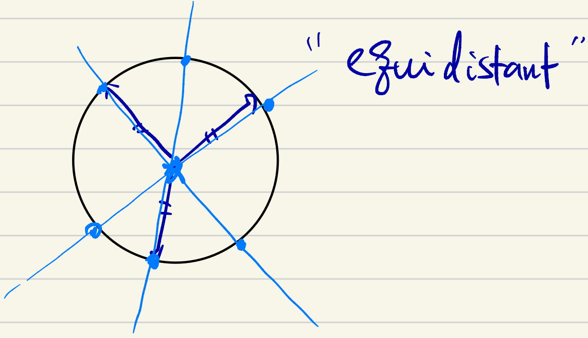
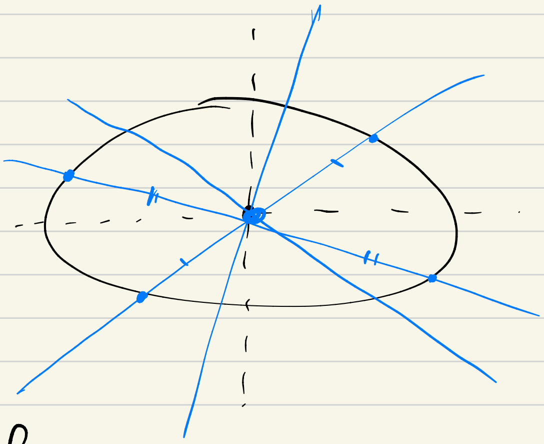
2025.8.7.

Lecture 2. Centers of conics, geometric transformations

1. The Concept of a Center

What is it?



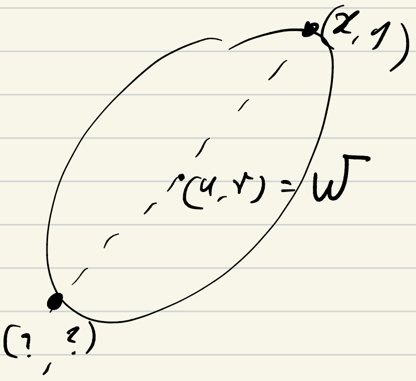


Def: Let $\omega = (u,v)$: fixed.

A Central reflection in ω is a mapping

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $(x,y) \longmapsto (2u-x, 2v-y)$

Note The mid point of (x,y) and its Central reflection in ω is ω itself.



Def: Let Q be a Conic.

A Center $\omega = (u,v)$ of Q is a point that Satisfies

$$Q(x,y) = Q(2u-x, 2v-y).$$

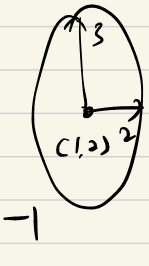
Example (1) For $Q(x,y) = (x-\alpha)^2 + (y-\beta)^2 + \gamma$

Verify that (α, β) is a center of Q.

$$(x-\alpha)^2 + (y-\beta)^2 + r \overset{\checkmark}{=} ((2\alpha - x) - \alpha)^2 + ((2\beta - y) - \beta)^2 + r$$

(2) $$Q(x,y) = \frac{(x-1)^2}{2^2} + \frac{(y-2)^2}{3^2} - 1$$

$$\overset{\vee}{=} \frac{((2\cdot 1 - x) - 1)^2}{2^2} + \frac{((2\cdot 2 - y) - 2)^2}{4^2} - 1$$



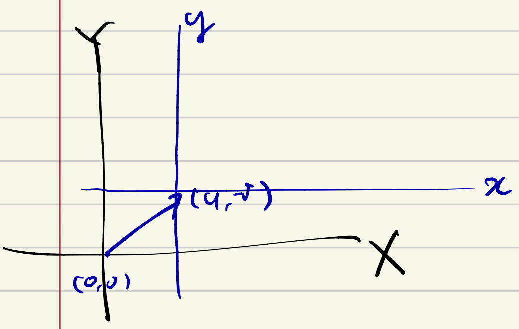
Note: This definition makes sense whether or not the zero set of a conic Q contains a point.

2. Finding Centers

Def: A translation of the plane through (u,v) is a mapping $\mathbb{R}^2 \to \mathbb{R}^2$

$$(X, Y) \mapsto (x, y)$$

where $x = X + u$
$y = Y + v$



Observation: Given a conic $Q(x,y)$ translated through (u,v) and obtain a new conic $R(X, Y) = Q(X+u, Y+v)$

Recall $Q(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

(1) The quadratic terms remain unchanged, while linear and constant terms vary.

(2) The constant term of $R(X,Y)$ is $Q(u,v)$

Example: $Q(x,y) = 2x^2+3y^2-12x+12y+24$ translated through $(3,-2)$

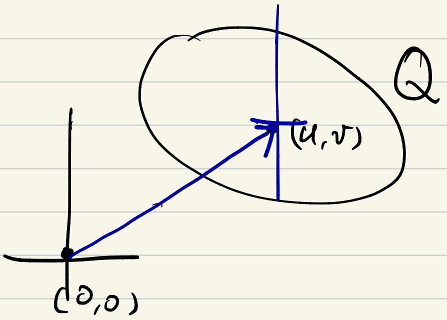
$$R(X,Y) = Q(X+3,\ Y-2)$$

$$= 2X^2+3Y^2+\underbrace{Q(3,-2)}_{=-6}.$$

Lemma: A point (u,v) is a center of $Q(x,y)$: conic

$\Longleftrightarrow$ $(0,0)$ is a center for the translated conic $R(X,Y) = Q(X+u, Y+v)$

Proof:



$$R(X,Y) = Q(X+u,\ Y+v)$$

$$\underset{\substack{\uparrow\\(u,v)\\\text{center of } Q}}{=} Q(2u-(x+u),\ 2v-(y+v))$$

$$= Q(u-x,\ v-y)$$

$$= R(-x,-y)$$

$$= R(2\cdot 0-x,\ 2\cdot 0-y)$$

$(0,0)$: center of R. □

Lemma: Let $Q(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$

$(0,0)$ is a center of Q $\Longleftrightarrow$ The coefficients of the linear terms x, y are both zero.

Proof: $(0,0)$ is a center of Q

$$\iff Q(x,y) = Q(-x,-y)$$

$\iff$ The following are identical

$$Q(x,y) = ax^2+2hxy+by^2 +2gx+2fy+c$$
$$Q(-x,-y) = ax^2+2hxy+by^2 -2gx-2fy+c$$

$$2g = -2g, \quad 2f = -2f$$

$$\therefore\ g = 0 \text{ and } f = 0. \qquad \blacksquare$$

Theorem (Center of a Conic) Let Q be as above. If (u,v) is a center of Q, then it is a solution to

$$\begin{cases} au + hv + g = 0 \\ hu + bv + f = 0 \end{cases}$$

and vice versa.

Proof: We've seen that

(u,v) : center of Q $\iff$ $(0,0)$ is a center of $Q(x+u, y+v)$

$\iff$ coefficients of linear terms of $Q(x+u, y+v)$ are zero.

$$Q(x+u, y+v) = a(x+u)^2 + 2h(x+u)(y+v) + b(y+v)^2 + 2g(x+u) + 2f(y+v) + c$$

$$= \underset{\sim\sim\sim}{\text{Quadratic}} + \underline{\text{Const}} + \overset{=0}{(2au + 2hv + 2g)}\,x$$

$$+ \overset{=0}{(2bv + 2hu + 2f)}\,y.$$

□

Note $Q(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

∂: "partial"

$$\frac{\partial Q}{\partial x}(u,v) = ax + hy + g \Big|_{(u,v)} = 0$$

$$\frac{\partial Q}{\partial y}(u,v) = hx + by + f \Big|_{(u,v)} = 0.$$

Example (1) $Q(x,y) = y^2 - 4ax$, $a>0$ standard parabola

Find center if any.

Sol

$$\frac{\partial Q}{\partial x} = -4\underline{a}^{>0} \cancel{=} 0.$$

$$\frac{\partial Q}{\partial y} = 2y = 0$$

Solution = center.

There is no center.

(2) $Q(x,y) = ax^2 + by^2 + c$

$$\frac{\partial Q}{\partial x} = 2ax = 0$$

$$\frac{\partial Q}{\partial y} = 2by = 0$$

If a, b both nonzero then $(x,y) = (0,0)$ is the only center.

If $c = 0$ $\Leftrightarrow$ The center $(0,0)$ is in the zero set of $Q(x,y)$

If $a=0$, $b\neq 0$, then any points on the straight line $y=0$ is a center. Similar for the case $a\neq 0$, $b=0$.

3. Geometry of Centers

Theorem: Let $Q(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$: Conic.

(1) Q has a unique center, a line of centers, or no center.

(2) Q has a unique center $\Longleftrightarrow$ $\delta\neq 0$.

(3) If Q has line of centers then $\Delta = 0$.

Proof (1) Obvious.

(2) $(*)\begin{cases} ax+hy+g=0 \\ hx+by+f=0 \end{cases}$ has a unique solution.

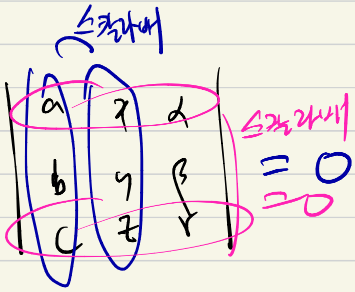
$\Longleftrightarrow$ $\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -g \\ -f \end{pmatrix}$ has a unique solution

$\Longleftrightarrow$ $\underset{\delta\,=}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}} \neq 0$

(3) $(*)$ has infinitely many solutions

$\Longleftrightarrow$ $\left.\begin{matrix}(a,h,g)\\(h,b,f)\end{matrix}\right)$ are of scalar multiple.

Exercise



$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$ ✓

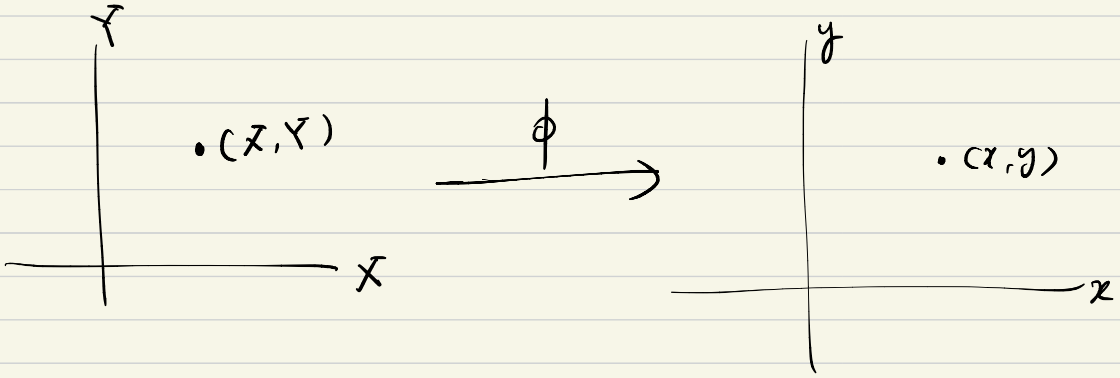
Scalar multiple

4. Congruences

Idea: In Geometry, (1) When can we say two objects are the same?

(2) Can you classify them?

Def: Consider a planar map ϕ s.t. $\phi(X,Y) = (x,y)$



Here $x = x(X,Y)$, $y = y(X,Y)$ are **components** of ϕ

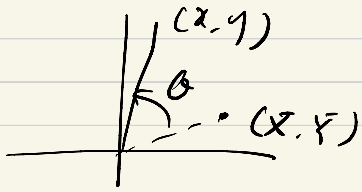
The map ϕ is **invertible** if for every (x,y) there is a unique (X,Y) such that $\phi(X,Y) = (x,y)$:

$$\phi^{-1}(x,y) = (X,Y)$$

The components of ϕ^{-1} are $X = X(x,y)$, $Y = Y(x,y)$

Def: The **rotation matrix** through an angle θ is

$$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Note

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\text{You check!}}{\overset{\downarrow}{=}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Note: $R(\theta)$ is a planar map.

Proposition: (1) $R(\theta_1)\circ R(\theta_2) \overset{\downarrow}{=} R(\theta_1+\theta_2) \overset{\downarrow}{=} R(\theta_2)\circ R(\theta_1)$

(2) $R(\theta)\circ R(-\theta) = I_2 = R(-\theta)\circ R(\theta)$

Notice that $\{ R(\theta) : \theta \in \mathbb{R} \}$ has a special algebraic feature

$$\left(\overset{\shortparallel}{SO(2)}, \cdot \right)$$

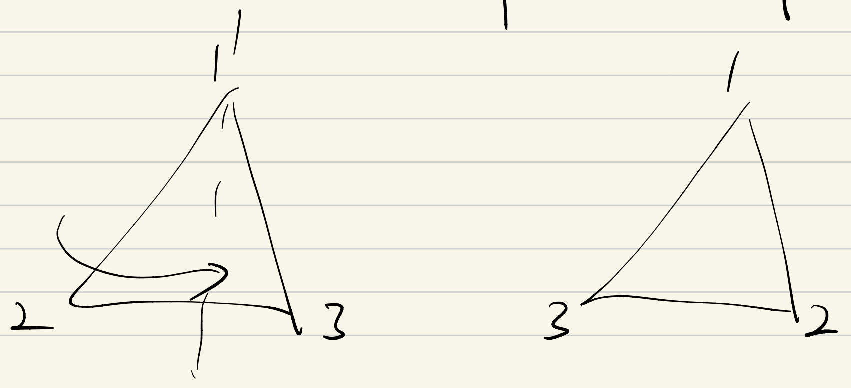
: (continuous) group (군 (이승)) Lie group

Def: A $\underset{}{\overset{\text{합동}}{\text{Congruence}}}$ is a planar map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$\phi(Z) = \underbrace{\boxed{R(\theta)}}_{\text{Rotational part}} Z + \underbrace{\boxed{T}}_{\text{Translational part}}$$

where $Z = (X, Y)$.

Remark: This definition of Congruence does not take into account possible reflections.



We can always write a congruence in terms of coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = R(\theta)\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{when } T = (u, v)$$

$$= \begin{cases} \cos\theta\, X - \sin\theta\, Y + u \\ \sin\theta\, X + \cos\theta\, Y + v . \end{cases}$$

We can also find the inverse of ϕ

$$\phi^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = R(-\theta)\begin{pmatrix} x \\ y \end{pmatrix} - R(-\theta)\overrightarrow{T}^{\,=\binom{u}{v}}$$

$$= \begin{cases} (x-u)\cos\theta + (y-v)\sin\theta \\ -(x-u)\sin\theta + (y-v)\cos\theta . \end{cases}$$

__Example__ Central reflection in the point (u, v)

$$(X, Y) \longrightarrow (x, y)$$

$$x = 2u - X$$
$$y = 2v - Y.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2u - X \\ 2v - Y \end{pmatrix} = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 2u \\ 2v \end{pmatrix}$$

$$\bullet : G \times G \to G$$
$$(a, b) \mapsto a \bullet b$$

__Def__: A pair $(G, \bullet)$ is a __group__ if (1) $\bullet$ is associative
(2) There is e such that $g \bullet e = e \cdot g = g$.

(3) For every $g \in G$, there is g^{-1}
Such that $gg^{-1} = e$.

Theorem The Congruences form a group.

$$\mathrm{Cong}(\mathbb{R}^2) = \{ \phi : \mathbb{R}^2 \to \mathbb{R}^2 : \phi : \text{congruence} \}$$

ϕ_1, ϕ_2, ϕ_3 : congruences

(1) $\phi_1(\phi_2 \phi_3) = (\phi_1 \phi_2)\phi_3$

(2) I_2 ✓

(3) ϕ^{-1} ? ✓