Thus $R_{imjk} = 0$ whenever the first two or last two indices are the same. Thus only four of the components are different from zero. They are

$$R_{1212} = R_{2121} = b_{22}b_{11} - b_{12}b_{21} = LN - M^2 = b \tag{10.34}$$

and

$$R_{1221} = R_{2112} = b_{12}b_{21} - b_{22}b_{11} = -(LN - M^2) = -b \tag{10.35}$$

Although the curvature tensors were defined in terms of the coefficients of the second fundamental form, they can in fact be expressed only in terms of the coefficients of the first fundamental form, i.e. the metric tensors, and their derivatives. In Problem 10.29, page 224, we prove

Theorem 10.9. $$R_{mijk} = \frac{\partial}{\partial u^j}\Gamma_{ikm} - \frac{\partial}{\partial u^k}\Gamma_{ijm} + \Gamma^{\alpha}_{ij}\Gamma_{mk\alpha} - \Gamma^{\alpha}_{ik}\Gamma_{mj\alpha}$$

Since the Christoffel symbols depend only upon the metric tensors and their derivatives, it follows that the same is true for the curvature tensors. We note that this is equivalent to Gauss's theorem, since from equation (10.34) the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{b}{g} = \frac{R_{1212}}{g}$$

Solved Problems

THEORY OF SURFACES

10.1. Show that the Gauss-Weingarten equations for a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ are

$$g\mathbf{x}_{uu} = pr\mathbf{x}_u + qr\mathbf{x}_v + rg^{1/2}\mathbf{N}$$
$$g\mathbf{x}_{uv} = ps\mathbf{x}_u + qs\mathbf{x}_v + sg^{1/2}\mathbf{N}$$
$$g\mathbf{x}_{vv} = pt\mathbf{x}_u + qt\mathbf{x}_v + tg^{1/2}\mathbf{N}$$

$$g^{3/2}\mathbf{N}_u = (spq - rq^2 - r)\mathbf{x}_u + (rpq - sp^2 - s)\mathbf{x}_v$$
$$g^{3/2}\mathbf{N}_v = (tpq - sq^2 - s)\mathbf{x}_u + (spq - tp^2 - t)\mathbf{x}_v$$

where $p = f_u,\ q = f_v,\ r = f_{uu},\ s = f_{uv},\ t = f_{vv},\ g = 1 + p^2 + q^2$.

$$\mathbf{x}_u = \mathbf{e}_1 + p\mathbf{e}_3,\quad \mathbf{x}_v = \mathbf{e}_2 + q\mathbf{e}_3,\quad \mathbf{x}_{uu} = r\mathbf{e}_3,\quad \mathbf{x}_{uv} = s\mathbf{e}_3,\quad \mathbf{x}_{vv} = t\mathbf{e}_3$$
$$E = \mathbf{x}_u\cdot\mathbf{x}_u = 1 + p^2,\quad F = \mathbf{x}_u\cdot\mathbf{x}_v = pq,\quad G = \mathbf{x}_v\cdot\mathbf{x}_v = 1 + q^2$$
$$EG - F^2 = 1 + p^2 + q^2 = g,\quad \mathbf{N} = \mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = -(p\mathbf{e}_1 + q\mathbf{e}_2 - \mathbf{e}_3)/g^{1/2}$$
$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = r/g^{1/2},\quad M = \mathbf{x}_{uv}\cdot\mathbf{N} = s/g^{1/2},\quad N = \mathbf{x}_{vv}\cdot\mathbf{N} = t/g^{1/2}$$
$$E_u = 2pr,\quad E_v = 2ps,\quad F_u = ps + qr,\quad F_v = pt + qs,\quad G_u = 2qs,\quad G_v = 2qt$$

From equations (10.2) and (10.4) we obtain

$$\Gamma^1_{11} = pr/g \qquad \Gamma^1_{12} = ps/g \qquad \Gamma^1_{22} = pt/g$$
$$\Gamma^2_{11} = qr/g \qquad \Gamma^2_{12} = qs/g \qquad \Gamma^2_{22} = qt/g$$

$$\beta^1_1 = (spq - rq^2 - r)/g^{3/2} \qquad \beta^1_2 = (tpq - sq^2 - s)/g^{3/2}$$
$$\beta^2_1 = (rpq - sp^2 - s)/g^{3/2} \qquad \beta^2_2 = (spq - tp^2 - t)/g^{3/2}$$

from which the result follows.

10.2. Using the Weingarten equations, show that

$$\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} \;=\; 0$$

where the third fundamental form $\mathrm{III} = d\mathbf{N}\cdot d\mathbf{N}$, H is the mean curvature and K is the Gaussian curvature.

Using equation *(10.2)*, page 201, we obtain

$$\begin{aligned}
\mathbf{N}_u\cdot\mathbf{N}_u &= (\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\cdot(\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\\
&= \frac{(MF-LG)^2E}{(EG-F^2)^2} + \frac{2(MF-LG)(LF-ME)F}{(EG-F^2)^2} + \frac{(LF-ME)^2G}{(EG-F^2)^2}\\
&= \frac{(-2LMF+L^2G+EM^2)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)L-(LN-M^2)E}{EG-F^2}\\
&= 2HL - KE
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbf{N}_u\cdot\mathbf{N}_v &= (\beta_1^1\mathbf{x}_u + \beta_1^2\mathbf{x}_v)\cdot(\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\\
&= \frac{(MF-LG)(NF-MG)E}{(EG-F^2)^2} + \frac{(NF-MG)(LF-ME)F}{(EG-F^2)^2}\\
&\quad + \frac{(MF-LG)(MF-NE)F}{(EG-F^2)^2} + \frac{(LF-ME)(MF-NE)G}{(EG-F^2)^2}\\
&= \frac{(MEN-M^2F+LGM-FLN)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)M}{EG-F^2} - \frac{(LN-M^2)F}{EG-F^2}\\
&= 2HM - KF
\end{aligned}$$

Also

$$\begin{aligned}
\mathbf{N}_v\cdot\mathbf{N}_v &= (\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\cdot(\beta_2^1\mathbf{x}_u + \beta_2^2\mathbf{x}_v)\\
&= \frac{(NF-MG)^2E}{(EG-F^2)^2} + \frac{2(NF-MG)(MF-NE)F}{(EG-F^2)^2} + \frac{(MF-NE)^2G}{(EG-F^2)^2}\\
&= \frac{(EN^2-2MFN+M^2G)(EG-F^2)}{(EG-F^2)^2} = \frac{(EN-2MF+LG)N}{EG-F^2} - \frac{(LN-M^2)G}{EG-F^2}\\
&= 2HN - KG
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{III} &= d\mathbf{N}\cdot d\mathbf{N} = (\mathbf{N}_u\,du + \mathbf{N}_v\,dv)\cdot(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) = \mathbf{N}_u\cdot\mathbf{N}_u\,du^2 + 2\mathbf{N}_u\cdot\mathbf{N}_v\,du\,dv + \mathbf{N}_v\cdot\mathbf{N}_v\,dv^2\\
&= (2HL-KE)\,du^2 + 2(2HM-KF)\,du\,dv + (2HN-KG)\,dv^2\\
&= 2H(L\,du^2 + 2M\,du\,dv + N\,dv^2) - K(E\,du^2 + 2F\,du\,dv + G\,dv^2) = 2H\,\mathrm{II} - K\,\mathrm{I}
\end{aligned}$$

which gives the required result.

10.3. Prove that the Christoffel symbols Γ_{ij}^k are given by equations *(10.4)*, page 202.

Observe that

$$\begin{aligned}
\mathbf{x}_u\cdot\mathbf{x}_{uu} &= \tfrac12(\mathbf{x}_u\cdot\mathbf{x}_u)_u = \tfrac12 E_u, & \mathbf{x}_u\cdot\mathbf{x}_{uv} &= \tfrac12(\mathbf{x}_u\cdot\mathbf{x}_u)_v = \tfrac12 E_v\\
\mathbf{x}_v\cdot\mathbf{x}_{vv} &= \tfrac12(\mathbf{x}_v\cdot\mathbf{x}_v)_v = \tfrac12 G_v, & \mathbf{x}_v\cdot\mathbf{x}_{uv} &= \tfrac12(\mathbf{x}_v\cdot\mathbf{x}_v)_u = \tfrac12 G_u
\end{aligned}$$

Also using the above,

$$\begin{aligned}
F_u &= (\mathbf{x}_u\cdot\mathbf{x}_v)_u = \mathbf{x}_{uu}\cdot\mathbf{x}_v + \mathbf{x}_u\cdot\mathbf{x}_{uv} = \mathbf{x}_{uu}\cdot\mathbf{x}_v + \tfrac12 E_v\\
F_v &= (\mathbf{x}_u\cdot\mathbf{x}_v)_v = \mathbf{x}_{uv}\cdot\mathbf{x}_v + \mathbf{x}_u\cdot\mathbf{x}_{vv} = \tfrac12 G_u + \mathbf{x}_u\cdot\mathbf{x}_{vv}
\end{aligned}$$

Hence

$$\mathbf{x}_v\cdot\mathbf{x}_{uu} = F_u - \tfrac12 E_v, \qquad \mathbf{x}_u\cdot\mathbf{x}_{vv} = F_v - \tfrac12 G_u$$

Now from the Gauss equations and the above,

$$\begin{aligned}
\tfrac12 E_u &= \mathbf{x}_u\cdot\mathbf{x}_{uu} = \Gamma_{11}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{11}^1 E + \Gamma_{11}^2 F\\
F_u - \tfrac12 E_v &= \mathbf{x}_v\cdot\mathbf{x}_{uu} = \Gamma_{11}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{11}^1 F + \Gamma_{11}^2 G\\
\tfrac12 E_v &= \mathbf{x}_u\cdot\mathbf{x}_{uv} = \Gamma_{12}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{12}^1 E + \Gamma_{12}^2 F\\
\tfrac12 G_u &= \mathbf{x}_v\cdot\mathbf{x}_{uv} = \Gamma_{12}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{12}^1 F + \Gamma_{12}^2 G\\
F_v - \tfrac12 G_u &= \mathbf{x}_u\cdot\mathbf{x}_{vv} = \Gamma_{22}^1\mathbf{x}_u\cdot\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_u\cdot\mathbf{x}_v = \Gamma_{22}^1 E + \Gamma_{22}^2 F\\
\tfrac12 G_v &= \mathbf{x}_v\cdot\mathbf{x}_{vv} = \Gamma_{22}^1\mathbf{x}_v\cdot\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_v\cdot\mathbf{x}_v = \Gamma_{22}^1 F + \Gamma_{22}^2 G
\end{aligned}$$

Solving the first two equations for Γ_{11}^1 and Γ_{11}^2, the second two equations for Γ_{12}^1 and Γ_{12}^2, and the last two equations for Γ_{22}^1 and Γ_{22}^2, we obtain

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \qquad \Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)} \qquad \Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma^2_{11} = \frac{2EF_u - EE_v + FE_u}{2(EG - F^2)} \qquad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

which are the required results.

10.4. Prove that $K(EG - F^2)^2 = [\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2$.

$$L = \mathbf{x}_{uu}\cdot\mathbf{N} = \mathbf{x}_{uu}\cdot\mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = [\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|$$

$$M = \mathbf{x}_{uv}\cdot\mathbf{N} = [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|, \qquad N = \mathbf{x}_{vv}\cdot\mathbf{N} = [\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v]/|\mathbf{x}_u\times\mathbf{x}_v|$$

Thus
$$LN - M^2 = \frac{[\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2}{|\mathbf{x}_u\times\mathbf{x}_v|^2}$$

Also
$$|\mathbf{x}_u\times\mathbf{x}_v|^2 = (\mathbf{x}_u\times\mathbf{x}_v)\cdot(\mathbf{x}_u\times\mathbf{x}_v) = (\mathbf{x}_u\cdot\mathbf{x}_u)(\mathbf{x}_v\cdot\mathbf{x}_v) - (\mathbf{x}_u\cdot\mathbf{x}_v)^2 = EG - F^2$$

Hence
$$K = \frac{LN - M^2}{EG - F^2} = \frac{[\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v][\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v] - [\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v]^2}{(EG - F^2)^2}$$

10.5. Using the result of the above problem, prove that

$$K(EG - F^2)^2 = (F_{uv} - \tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu})(EG - F^2) + \det\begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

Note that this is a direct proof of Gauss' Theorem.

We note that

$$[\mathbf{abc}][\mathbf{def}] = \det\begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{pmatrix}\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$= \det\left(\begin{bmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{bmatrix}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right) = \det\begin{pmatrix} \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{d} \\ \mathbf{a}\cdot\mathbf{e} & \mathbf{b}\cdot\mathbf{e} & \mathbf{c}\cdot\mathbf{e} \\ \mathbf{a}\cdot\mathbf{f} & \mathbf{b}\cdot\mathbf{f} & \mathbf{c}\cdot\mathbf{f} \end{pmatrix}$$

Thus from Problem 10.4 and the computations from Problem 10.3, we have

$$K(EG - F^2)^2 = \det\begin{pmatrix} \mathbf{x}_{uu}\cdot\mathbf{x}_{vv} & \mathbf{x}_u\cdot\mathbf{x}_{vv} & \mathbf{x}_v\cdot\mathbf{x}_{vv} \\ \mathbf{x}_{uu}\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_u \\ \mathbf{x}_{uu}\cdot\mathbf{x}_v & \mathbf{x}_u\cdot\mathbf{x}_v & \mathbf{x}_v\cdot\mathbf{x}_v \end{pmatrix} - \det\begin{pmatrix} \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} & \mathbf{x}_u\cdot\mathbf{x}_{uv} & \mathbf{x}_v\cdot\mathbf{x}_{uv} \\ \mathbf{x}_{uv}\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_u \\ \mathbf{x}_{uv}\cdot\mathbf{x}_v & \mathbf{x}_u\cdot\mathbf{x}_v & \mathbf{x}_v\cdot\mathbf{x}_v \end{pmatrix}$$

$$= \det\begin{pmatrix} \mathbf{x}_{uu}\cdot\mathbf{x}_{vv} & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

Since both determinants have the common minor $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$, it follows that

$$K(EG - F^2)^2 = (\mathbf{x}_{uu}\cdot\mathbf{x}_{vv} - \mathbf{x}_{uv}\cdot\mathbf{x}_{uv})(EG - F^2) + \det\begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

In Problem 10.3 we obtained $\mathbf{x}_{uu}\cdot\mathbf{x}_v = F_u - \frac{1}{2}E_v$ and $\mathbf{x}_{uv}\cdot\mathbf{x}_v = \frac{1}{2}G_u$. Hence

$$(F_u - \tfrac{1}{2}E_v)_v = (\mathbf{x}_{uu}\cdot\mathbf{x}_v)_v = \mathbf{x}_{uuv}\cdot\mathbf{x}_v + \mathbf{x}_{uu}\cdot\mathbf{x}_{vv}, \qquad (\tfrac{1}{2}G_u)_u = (\mathbf{x}_{uv}\cdot\mathbf{x}_v)_u = \mathbf{x}_{uvu}\cdot\mathbf{x}_v + \mathbf{x}_{uv}\cdot\mathbf{x}_{uv}$$

Subtracting,

$$\mathbf{x}_{uu}\cdot\mathbf{x}_{vv} - \mathbf{x}_{uv}\cdot\mathbf{x}_{uv} = (F_u - \tfrac{1}{2}E_v)_v - \tfrac{1}{2}G_{uu} = F_{uv} - \tfrac{1}{2}E_{vv} - \tfrac{1}{2}G_{uu}$$

which gives the required result.

10.6. If the parameter curves on a patch are lines of curvature, show that the Codazzi-Mainardi equations (*10.7*) take the form

$$\frac{\partial\kappa_1}{\partial v} = \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1), \qquad \frac{\partial\kappa_2}{\partial u} = \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)$$

where κ_1 and κ_2 are the principal curvatures.

When the parameter curves are lines of curvature, $F = M = 0$. Thus equations (*10.7*) reduce to

$$L_v = L\Gamma^1_{12} - N\Gamma^2_{11} = \frac{LGE_v}{2EG} + \frac{NEE_v}{2EG} = \tfrac{1}{2}E_v\left(\frac{L}{E} + \frac{N}{G}\right)$$

and

$$N_u = -L\Gamma^1_{22} + N\Gamma^2_{12} = \frac{LGG_u}{2EG} + \frac{NEG_u}{2EG} = \tfrac{1}{2}G_u\left(\frac{L}{E} + \frac{N}{G}\right)$$

or

$$\left(\frac{L}{E}\right)_v = \frac{E_v}{2E}\left(\frac{N}{G} - \frac{L}{E}\right) \quad \text{and} \quad \left(\frac{N}{G}\right)_u = \frac{G_u}{2G}\left(\frac{L}{E} - \frac{N}{G}\right)$$

But from Theorem 9.13, page 186, if the parameter curves are lines of curvature, $\kappa_1 = L/E$ and $\kappa_2 = N/G$, which gives the required result.

10.7. Prove that there does not exist a compact surface in E^3 of class $\geqq 2$ with Gaussian curvature $K \leqq 0$.

Suppose otherwise, i.e. suppose S is a compact surface of class $\geqq 2$ with $K \leqq 0$ at each point. Now consider the real-valued function $f(P) = |\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$, where $\mathbf{x}$ is the point P. We leave as an exercise for the reader to show that f is continuous on S. Hence from Theorem 6.9, page 110, f takes on its maximum, say, $f(P_0) = |\mathbf{x}_0|^2 = r^2$ at some point P_0 on S. Note that $r^2 > 0$. Otherwise, we would have $f \equiv 0$ on S, since $f \geqq 0$ and r^2 is its maximum. But then S would consist of a single point $\mathbf{x} = \mathbf{0}$, which is impossible. Now let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on S containing P_0 such that the u and v parameter curves have principal directions at P_0. Since $f(P) = f(\mathbf{x}(u,v))$ has a maximum at P_0,

$$\partial f/\partial u = 2\mathbf{x}\cdot\mathbf{x}_u = 0 \quad \text{and} \quad \partial f/\partial v = 2\mathbf{x}\cdot\mathbf{x}_v = 0$$

at P_0. Also

$$\partial^2 f/\partial u^2 = 2\mathbf{x}_u\cdot\mathbf{x}_u + 2\mathbf{x}\cdot\mathbf{x}_{uu} \leqq 0 \quad \text{and} \quad \partial^2 f/\partial v^2 = 2\mathbf{x}_v\cdot\mathbf{x}_v + 2\mathbf{x}\cdot\mathbf{x}_{vv} \leqq 0$$

at P_0. From the first two equations above we obtain that $\mathbf{x}$ is orthogonal to $\mathbf{x}_u$ and $\mathbf{x}_v$ at P_0. Hence $\mathbf{N} = \pm\mathbf{x}/|\mathbf{x}| = \pm\mathbf{x}/r$ at P_0. We may assume that the sense of $\mathbf{N}$ is such that $\mathbf{N} = \mathbf{x}/r$. Substituting into the second two equations above, we obtain that $\mathbf{x}_u\cdot\mathbf{x}_u + r\mathbf{N}\cdot\mathbf{x}_{uu} \leqq 0$ and $\mathbf{x}_v\cdot\mathbf{x}_v + r\mathbf{N}\cdot\mathbf{x}_{vv} \leqq 0$, or $E + rL \leqq 0$ and $G + rN \leqq 0$, or $L/E \leqq -1/r < 0$ and $N/G \leqq -1/r < 0$ at P_0. Since the u and v parameter curves have principal directions at P_0, it follows from Theorem 9.11, page 185, that $\kappa_1 = L/E$ and $\kappa_2 = N/G$, and hence from the above $K = \kappa_1\kappa_2 = LN/EG \geqq 1/r^2 > 0$ at P_0, which is again impossible since $K \leqq 0$ on S. Thus the proposition is proved.

10.8. Prove that $f(P) = [\kappa_1(P) - \kappa_2(P)]^2$ is a continuous function on a surface.

Recall that the principal curvatures at a point P on a surface S depends on the orientation of the patch containing P, changing sign when there is a change in the sense of $\mathbf{N}$. Thus unless S is orientable, it may not be possible to define $\kappa_1(P)$ and $\kappa_2(P)$ themselves as continuous functions throughout S. Note however that f is independent of a change in sign of both κ_1 and κ_2 and hence is an intrinsic property of S, independent of the patch containing P.

To prove that f is continuous at a point P_0, we suppose that $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P_0. Since κ_1 and κ_2 are continuous functions of the first and second fundamental coefficients, $f(P) = f(\mathbf{x}(u,v))$ is a continuous function of u and v. Thus given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|f(\mathbf{x}(u,v)) - f(\mathbf{x}(u_0,v_0))| < \epsilon$ for (u,v) in $S_{\delta_1}(u_0,v_0)$. From Problem 8.13, page 165, the image M of $S_{\delta_1}(u_0,v_0)$ on S is the intersection of an open set O in E^3 with S. It follows that there exists an $S_\delta(\mathbf{x}_0)$ in E^3 such that $S_\delta(\mathbf{x}_0) \cap S$ is contained in M. Thus for $\mathbf{x}$ in $S_\delta(\mathbf{x}_0) \cap S$ we have $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. This shows that f is continuous at P_0, which proves the proposition.

10.9. Prove *Hilbert's lemma*: If at a point P_0 on a surface of sufficiently high class, (i) $\kappa_1(P_0)$ is a local maximum, (ii) $\kappa_2(P_0)$ local minimum, (iii) $\kappa_1(P_0) > \kappa_2(P_0)$, then $K(P_0) \leqq 0$.

Since $\kappa_1(P_0) \neq \kappa_2(P_0)$, P_0 is not an umbilical point. Thus from Theorem 9.10, page 185, there exists a patch $\mathbf{x} = \mathbf{x}(u,v)$ containing P_0 for which the parameter curves are lines of curvatures. From Problem 10.6 it follows that

$$\frac{\partial \kappa_1}{\partial v} = \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial u} = \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)$$

Differentiating,
$$\frac{\partial^2 \kappa_1}{\partial v^2} = \frac{1}{2}\left(\frac{EE_{vv} - E_v^2}{E^2}\right)(\kappa_2 - \kappa_1) + \frac{1}{2}\frac{E_v}{E}(\kappa_2 - \kappa_1)_v$$

$$\frac{\partial^2 \kappa_2}{\partial u^2} = \frac{1}{2}\left(\frac{GG_{uu} - G_u^2}{G^2}\right)(\kappa_1 - \kappa_2) + \frac{1}{2}\frac{G_u}{G}(\kappa_1 - \kappa_2)_u$$

Since κ_1 and κ_2 are extreme values at P_0, $\partial\kappa_1/\partial v = \partial\kappa_2/\partial u = 0$ at P_0. Also $\kappa_1 \neq \kappa_2$ at P_0. Thus from the first two equations above, $E_v = G_u = 0$ at P_0. Substituting into the second two equations above, it follows that

$$\frac{\partial^2 \kappa_1}{\partial v^2} = \frac{1}{2}\frac{E_{vv}}{E}(\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial u^2} = \frac{1}{2}\frac{G_{uu}}{G}(\kappa_1 - \kappa_2)$$

Since κ_1 is a maximum at P_0, $\partial^2\kappa_1/\partial v^2 \leqq 0$ at P_0. Also $\kappa_1 > \kappa_2$ at P_0 and $E > 0$. Hence from the first equation above, $E_{vv} \geqq 0$ at P_0. Since κ_2 is a minimum at P_0, $\partial^2\kappa_2/\partial u^2 \geqq 0$ at P_0. Also $G > 0$. Hence $G_{uu} \geqq 0$ at P_0. Finally, since the parameter curves are lines of curvature, we have $F = M = 0$. Also at P_0, $E_v = 0$ and $G_u = 0$. From Problem 10.5 it follows that at P_0

$$K = -\frac{1}{2}\frac{E_{vv} + G_{uu}}{EG}$$

Since $E_{vv} \geqq 0$ and $G_{uu} \geqq 0$ it follows that $K \leqq 0$, which is the required result.

10.10. Prove Theorem 10.7: The only connected and compact surfaces of sufficiently high class with constant Gaussian curvature are spheres.

Suppose S is a connected and compact surface with $K = \text{constant}$. From Problem 10.7, not every point on S can have $K \leqq 0$. Hence we can assume $K = \text{constant} > 0$. Now if we can show that every point on S is a spherical umbilical point, then it would follow from Theorem 10.5, page 205, that S is a sphere and we are finished. In order to show that every point on S is a spherical umbilical point, we consider the function $f(P) = [(\kappa_1(P) - \kappa_2(P)]^2$. From Problem 10.8, $f(P)$ is continuous on S. Since S is compact, f takes on an absolute maximum at some point P_0 on S. Now suppose $f > 0$ at P_0. Since f is continuous at P_0, $f > 0$ in some neighborhood $S(P_0)$. Since $f = (\kappa_1 - \kappa_2)^2 > 0$ in $S(P_0)$, $\kappa_1 \neq \kappa_2$ in $S(P_0)$. Also κ_1 and κ_2 have the same sign in $S(P_0)$ since $K = \kappa_1\kappa_2 > 0$ in $S(P_0)$. Thus we can assume $\kappa_1 > \kappa_2 > 0$ in $S(P_0)$. Since $\kappa_1 - \kappa_2 > 0$ in $S(P_0)$ and $(\kappa_1 - \kappa_2)^2$ has a maximum at P_0, it follows that $\kappa_1 - \kappa_2$ has a local maximum at P_0. Since $K = \kappa_1\kappa_2 = \text{constant} > 0$, κ_2 decreases when κ_1 increases and it follows that κ_1 has a local maximum at P_0 and κ_2 has a local minimum at P_0. Thus we see that if $f > 0$ at P_0, then (i) κ_1 has a local maximum at P_0, (ii) κ_2 has a local minimum at P_0 and (iii) $\kappa_1 > \kappa_2$ at P_0. It follows from Problem 10.9 that $K \leqq 0$ at P_0. But this is impossible since $K > 0$ on S. Thus f is not positive at P_0. But f takes on its maximum at P_0 and $f(P) \geqq 0$ for all P. Hence $f \equiv 0$ on S. It follows that $\kappa_1 = \kappa_2$ at each P on S. Since the principal curvatures are extreme values of the normal curvature at P and since $K > 0$, it follows that the normal curvature $\kappa = \text{constant} \neq 0$ at each P. Namely, every point on S is a spherical umbilical point and hence S is a sphere.

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10.11. If $v^i = a^i_\alpha u^\alpha$ and $w^i = b^i_\alpha v^\alpha$, show that $w^i = b^i_\alpha a^\alpha_\beta u^\beta$.

We write $v^\alpha = \sum_\beta a^\alpha_\beta u^\beta$. Hence

$$w^i = \sum_\alpha b^i_\alpha v^\alpha = \sum_\alpha b^i_\alpha \sum_\beta a^\alpha_\beta u^\beta = \sum_\alpha \sum_\beta b^i_\alpha a^\alpha_\beta u^\beta = b^i_\alpha a^\alpha_\beta u^\beta$$

10.12. Show that $g_{i\alpha} g^{\alpha j} = \delta^j_i$, $\alpha, i, j = 1, 2$, where the $g^{\alpha j}$ are defined in equation *(10.11)*.

$$g_{1\alpha} g^{\alpha 1} = g_{11} g^{11} + g_{12} g^{21} = g_{11} g_{22}/g - g_{12} g_{12}/g = g/g = 1$$

$$g_{1\alpha} g^{\alpha 2} = g_{11} g^{12} + g_{12} g^{22} = -g_{11} g_{12}/g + g_{12} g_{11}/g = 0$$

$$g_{2\alpha} g^{\alpha 1} = g_{21} g^{11} + g_{22} g^{21} = g_{21} g_{22}/g - g_{22} g_{12}/g = 0$$

$$g_{2\alpha} g^{\alpha 2} = g_{21} g^{12} + g_{22} g^{22} = -g_{21} g_{21}/g + g_{22} g_{11}/g = g/g = 1$$

Hence

$$g_{i\alpha} g^{\alpha j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \delta^j_i$$

10.13. Show that $\delta^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$, where the δ^{pq}_{ij} are defined in Example 10.7(*b*), page 212.

Let $A^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$. Clearly $A^{pq}_{ij} = 0$ if $i = j$. Now suppose $i \neq j$ and $p \neq i$. Then $\delta^p_i = 0$ and $A^{pq}_{ij} = 0$ unless $p = j$ and $q = i$, in which case $A^{pq}_{ij} = -1$. If $i \neq j$ and $p = i$, then $\delta^p_j = 0$. Hence $A^{pq}_{ij} = 0$ unless also $q = j$, in which case $A^{pq}_{ij} = 1$. Thus

$$A^{pq}_{ij} = \left\{ \begin{array}{rl} 1 & \text{if } i \neq j,\ p = i,\ q = j \\ -1 & \text{if } i \neq j,\ q = i,\ p = j \\ 0 & \text{otherwise} \end{array} \right\} = \delta^{pq}_{ij}$$

which is the required result.

10.14. Show that the δ^{pq}_{ij} in the above problem are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

$$\begin{aligned}
\delta^{\gamma\sigma}_{\alpha\beta} \frac{\partial \bar{u}^p}{\partial u^\gamma} \frac{\partial \bar{u}^q}{\partial u^\sigma} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} &= (\delta^\gamma_\alpha \delta^\sigma_\beta - \delta^\gamma_\beta \delta^\sigma_\alpha) \frac{\partial \bar{u}^p}{\partial u^\gamma} \frac{\partial \bar{u}^q}{\partial u^\sigma} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \\
&= \left(\delta^\gamma_\alpha \frac{\partial \bar{u}^p}{\partial u^\gamma}\right)\left(\delta^\sigma_\beta \frac{\partial \bar{u}^q}{\partial u^\sigma}\right)\left(\frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) - \left(\delta^\gamma_\beta \frac{\partial \bar{u}^p}{\partial u^\gamma}\right)\left(\delta^\sigma_\alpha \frac{\partial \bar{u}^q}{\partial u^\sigma}\right)\left(\frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) \\
&= \frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} - \frac{\partial \bar{u}^p}{\partial u^\beta} \frac{\partial \bar{u}^q}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \\
&= \left(\frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i}\right)\left(\frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial u^\beta}{\partial \bar{u}^j}\right) - \left(\frac{\partial \bar{u}^p}{\partial u^\beta} \frac{\partial u^\beta}{\partial \bar{u}^j}\right)\left(\frac{\partial \bar{u}^q}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i}\right) \\
&= \delta^p_i \delta^q_j - \delta^p_j \delta^q_i = \delta^{pq}_{ij} = \bar{\delta}^{pq}_{ij}
\end{aligned}$$

Thus δ^{pq}_{ij} are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

Another Method. The products $\delta^p_i \delta^q_j$ and $\delta^p_j \delta^q_i$ are the outer products of mixed absolute tensors covariant of order 1 and contravariant of order 1. Hence they are the components of absolute tensors covariant of order 2 and contravariant of order 2. It follows that the differences

$$\delta^{pq}_{ij} = \delta^p_i \delta^q_j - \delta^p_j \delta^q_i$$

are the components of an absolute tensor covariant of order 2 and contravariant of order 2.

10.15. Show that the contraction A^α_α of the components A^j_i of an absolute mixed tensor is a scalar invariant.

Since $\bar{A}^j_i = A^\beta_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial \bar{u}^j}{\partial u^\beta}$, we have $\bar{A}^\gamma_\gamma = A^\beta_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^\gamma}\frac{\partial \bar{u}^\gamma}{\partial u^\beta}$. From equation *(10.16)*, page 208, $\frac{\partial u^\alpha}{\partial \bar{u}^\gamma}\frac{\partial \bar{u}^\gamma}{\partial u^\beta} = \delta^\alpha_\beta$. Hence $\bar{A}^\gamma_\gamma = A^\beta_\alpha \delta^\alpha_\beta = A^\alpha_\alpha$, which is the required result.

10.16. If $\bar{A}^{pq}_{ij}$ are the components of a tensor contravariant of order 2, covariant of order 2, and of weight N, show that the contraction $A^{\alpha q}_{\alpha j}$ are the components of a mixed tensor contravariant of order 1, covariant of order 1, and of weight N.

Since $\bar{A}^{pq}_{ij} = \left[\det\left(\frac{\partial u_i}{\partial \bar{u}_j}\right)\right]^N A^{\gamma\sigma}_{\alpha\beta}\frac{\partial \bar{u}^p}{\partial u^\gamma}\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}$, we have

$$\bar{A}^{nq}_{nj} = \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A^{\gamma\sigma}_{\alpha\beta}\frac{\partial \bar{u}^n}{\partial u^\gamma}\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\alpha}{\partial \bar{u}^n}\frac{\partial u^\beta}{\partial \bar{u}^j}$$

$$= \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A^{\gamma\sigma}_{\alpha\beta}\delta^\alpha_\gamma\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\beta}{\partial \bar{u}^j} = \left[\det\left(\frac{\partial u_i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha\sigma}_{\alpha\beta}\frac{\partial \bar{u}^q}{\partial u^\sigma}\frac{\partial u^\beta}{\partial \bar{u}^j}$$

That is, the $A^{\alpha q}_{\alpha j}$ transform as an absolute mixed tensor of weight N, which is the required result.

10.17. If $\bar{u}^i = \bar{u}^i(u^1,\ldots,u^n)$, $i = 1,\ldots,n$, is an allowable coordinate transformation on a coordinate manifold of n dimensions and $u^i = u^i(\bar{u}^1,\ldots,\bar{u}^n)$ is its inverse, show that $\frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta^j_i$.

From the chain rule,

$$\frac{\partial \bar{u}^j}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^1}\frac{\partial u^1}{\partial \bar{u}^i} + \frac{\partial u^j}{\partial u^2}\frac{\partial u^2}{\partial \bar{u}^i} + \cdots + \frac{\partial \bar{u}^j}{\partial u^n}\frac{\partial u^n}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}$$

But $\frac{\partial \bar{u}^j}{\partial \bar{u}^i} = \begin{Bmatrix} 1 & \text{if } i = j \\ 0 & \text{otherwise}\end{Bmatrix} = \delta^j_i$. Thus $\frac{\partial \bar{u}^j}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta^j_i$.

10.18. If $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ and $B^{i_1\cdots i_r}_{j_1\cdots j_s}$ are the components of two tensors A and B, contravariant and covariant of the same orders and of the same weight, show that

$$C^{i_1\cdots i_r}_{j_1\cdots j_s} = A^{i_1\cdots i_r}_{j_1\cdots j_s} + B^{i_1\cdots i_r}_{j_1\cdots j_s}$$

are the components of a tensor, contravariant and covariant of the same orders and of the same weight as A and B.

$$\bar{C}^{i_1\cdots i_r}_{j_1\cdots j_s} = \bar{A}^{i_1\cdots i_r}_{j_1\cdots j_s} + \bar{B}^{i_1\cdots i_r}_{j_1\cdots j_s}$$

$$= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} + \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N B^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

$$= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N [A^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} + B^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}]\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

$$= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N C^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

which is the required result.

10.19. If the components $A^{i_1\cdots i_r}_{j_1\cdots j_s}$ of a tensor are symmetric with respect to, say, the i_1 and i_2 indices, show that the transformed components

$$\bar{A}^{i_1\cdots i_r}_{j_1\cdots j_s} = \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}\frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

are also symmetric with respect to i_1 and i_2.

$$\bar{A}^{i_2 i_1 \cdots i_r}_{j_1 \cdots j_s} = \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \frac{\partial \bar{u}^{i_2}}{\partial u^{\alpha_1}} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_2}} \cdots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}}$$

$$= \left[\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)\right]^N A^{\alpha_2 \alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_2}} \frac{\partial \bar{u}^{i_2}}{\partial u^{\alpha_1}} \cdots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} = A^{i_1 i_2 \cdots i_r}_{j_1 \cdots j_s}$$

10.20. If the quantity $J = C^{ij}A_iB_j$ is a scalar invariant with respect to the components A_i and B_j of any two covariant vectors, show that the C^{ij} are the components of an absolute contravariant tensor of rank 2.

It is given that the sum $C^{ij}A_iB_j = \bar{C}^{\alpha\beta}\bar{A}_\alpha\bar{B}_\beta$ for any A_i and B_j. Hence

$$C^{ij}A_iB_j = \bar{C}^{\alpha\beta}\bar{A}_\alpha\bar{B}_\beta = \bar{C}^{\alpha\beta}A_\gamma\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}B_\sigma\frac{\partial u^\sigma}{\partial \bar{u}^\beta} = \bar{C}^{\alpha\beta}\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\frac{\partial u^\sigma}{\partial \bar{u}^\beta}A_\gamma B_\sigma$$

Identifying coefficients gives $C^{ij} = \bar{C}^{\alpha\beta}\dfrac{\partial u^i}{\partial \bar{u}^\alpha}\dfrac{\partial u^j}{\partial \bar{u}^\beta}$ which shows that the C^{ij} are the components of an absolute tensor of rank 2.

10.21. If A_{ij} and B_{ij} are the components of symmetric tensors, and if x_i and y_i are components of contravariant vectors such that

$$(A_{ij} - \kappa_1 B_{ij})x^i = 0, \quad (A_{ij} - \kappa_2 B_{ij})y^i = 0, \qquad i, j = 1, \ldots, n, \ \kappa_1 \neq \kappa_2$$

show that $A_{ij}x^ix^j = B_{ij}x^ix^j = 0$ and that κ_1 is a scalar invariant.

Since $(A_{ij} - \kappa_1 B_{ij})x^i = 0$ for all j, we have $(A_{ij} - \kappa_1 B_{ij})x^iy^j = 0$. Similarly from the second equation $(A_{ij} - \kappa_2 B_{ij})y^ix^j = 0$, or, since the A_{ij} and B_{ij} are symmetric, $(A_{ij} - \kappa_2 B_{ij})x^iy^j = 0$. Subtracting, we obtain $(\kappa_1 - \kappa_2)B_{ij}x^iy^j = 0$. Since $\kappa_1 \neq \kappa_2$, it follows that $B_{ij}x^iy^j = 0$ and hence $A_{ij}x^iy^j = 0$. To show that κ_1 is a scalar invariant, we suppose that $\bar{z}^i$ are the components of an arbitrary contravariant vector and we consider the sum

$$(\bar{A}_{ij} - \kappa_1\bar{B}_{ij})\bar{x}^i\bar{z}^j = (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}x^\gamma\frac{\partial \bar{u}^i}{\partial u^\gamma}z^\sigma\frac{\partial \bar{u}^j}{\partial u^\sigma}$$

$$= (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\gamma z^\sigma\left(\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial \bar{u}^i}{\partial u^\gamma}\right)\left(\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial \bar{u}^j}{\partial u^\sigma}\right)$$

$$= (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\gamma z^\sigma\delta^\alpha_\gamma\delta^\beta_\sigma = (A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\alpha z^\beta$$

Since $(A_{\alpha\beta} - \kappa_1 B_{\alpha\beta})x^\alpha = 0$ for all β, $(\bar{A}_{ij} - \kappa_1\bar{B}_{ij})\bar{x}^i\bar{z}^j = 0$. But the $\bar{z}^i$ are arbitrary. Hence $(\bar{A}_{ij} - \kappa_1\bar{B}_{ij})\bar{x}^i = 0$ for all j. Thus κ_1 is a scalar invariant.

APPLICATIONS OF TENSORS

10.22. Show that the components du^i, $i = 1, 2$, of a tangent vector $d\mathbf{x} = \mathbf{x}_\alpha\, du^\alpha$ transform as the components of a contravariant vector. They are called the *contravariant components* of $d\mathbf{x}$.

Suppose $\bar{u}^i = \bar{u}^i(u^1, u^2)$, $i = 1, 2$, is an allowable parameter transformation with inverse $u^i = u^i(\bar{u}^1, \bar{u}^2)$, $i = 1, 2$. Then from the chain rule, $\mathbf{x}_\alpha = \dfrac{\partial \mathbf{x}}{\partial u^\alpha} = \dfrac{\partial \mathbf{x}}{\partial \bar{u}^1}\dfrac{\partial \bar{u}^1}{\partial u^\alpha} + \dfrac{\partial \mathbf{x}}{\partial \bar{u}^2}\dfrac{\partial \bar{u}^2}{\partial u^\alpha} = \dfrac{\partial \mathbf{x}}{\partial \bar{u}^i}\dfrac{\partial \bar{u}^i}{\partial u^\alpha}$. It follows that $d\mathbf{x} = \mathbf{x}_\alpha\, du^\alpha = \dfrac{\partial \mathbf{x}}{\partial \bar{u}^i}\dfrac{\partial \bar{u}^i}{\partial u^\alpha}du^\alpha = \dfrac{\partial \mathbf{x}}{\partial \bar{u}^i}d\bar{u}^i$. Hence $d\bar{u}^i = du^\alpha\dfrac{\partial \bar{u}^i}{\partial u^\alpha}$, which is the required result.

10.23. Show that $\dfrac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$.

Differentiating $g_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ with respect to u^k, $\partial g_{ij}/\partial u^k = \mathbf{x}_{ik} \cdot \mathbf{x}_j + \mathbf{x}_i \cdot \mathbf{x}_{jk}$. By definition, $\Gamma_{ijk} = \mathbf{x}_{ij} \cdot \mathbf{x}_k$. Hence $\partial g_{ij}/\partial u^k = \Gamma_{ikj} + \Gamma_{jki}$.

10.24. Show that $\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right]$.

From the above problem, $\frac{\partial g_{jk}}{\partial u^i} = \Gamma_{jik} + \Gamma_{kij}$, $\frac{\partial g_{ki}}{\partial u^j} = \Gamma_{kji} + \Gamma_{ijk}$, and $\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$. Since $\Gamma_{ijk} = \Gamma_{jik}$ for all i, j, k, it follows that $\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2\Gamma_{ijk}$.

10.25. Show that $\frac{\partial g}{\partial u^i} = 2g\Gamma^\alpha_{\alpha i}$.

$$\frac{\partial g}{\partial u^i} = \frac{\partial}{\partial u^i}(g_{11}g_{22} - (g_{12})^2) = \frac{\partial g_{11}}{\partial u^i}g_{22} + g_{11}\frac{\partial g_{22}}{\partial u^i} - 2g_{12}\frac{\partial g_{12}}{\partial u^i}$$
$$= g\left[g^{11}\frac{\partial g_{11}}{\partial u^i} + g^{22}\frac{\partial g_{22}}{\partial u^i} + 2g^{12}\frac{\partial g_{12}}{\partial u^i}\right] = gg^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial u^i}$$

where we used equation *(10.11)*, page 206. From Problem 10.23,

$$\frac{\partial g}{\partial u^i} = gg^{\alpha\beta}(\Gamma_{\alpha i\beta} + \Gamma_{\beta i\alpha}) = g(g^{\alpha\beta}\Gamma_{\alpha i\beta} + g^{\alpha\beta}\Gamma_{\beta i\alpha}) = g(\Gamma^\alpha_{\alpha i} + \Gamma^\beta_{\beta i}) = 2g\Gamma^\alpha_{\alpha i}$$

where we used $\Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij\alpha}$, and where we replaced the dummy index β by α.

10.26. Show that $R_{mijk} = g_{\alpha m}R^\alpha_{ijk}$,

From *(10.31)*, page 214, $g_{\alpha m}R^\alpha_{ijk} = g_{\alpha m}g^{\beta\alpha}R_{\beta ijk} = \delta^\beta_m R_{\beta ijk} = R_{mijk}$.

10.27. Show that the Christoffel symbols of the first kind transform in accordance with the law

$$\bar{\Gamma}_{ijk} = \left\{\Gamma_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\right\}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

We recall that the g_{jk} are the components of a covariant tensor of rank 2. Thus $\bar{g}_{jk} = g_{\beta\gamma}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k}$. Differentiating with respect to $\bar{u}^i$,

$$\frac{\partial \bar{g}_{jk}}{\partial \bar{u}^i} = \frac{\partial g_{\beta\gamma}}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\beta\gamma}\frac{\partial^2 u^\beta}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\beta\gamma}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial^2 u^\gamma}{\partial \bar{u}^i\,\partial \bar{u}^k}$$
$$= \frac{\partial g_{\beta\gamma}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^j}$$

where we used the chain rule $\frac{\partial g_{\beta\gamma}}{\partial \bar{u}^i} = \frac{\partial g_{\beta\gamma}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial \bar{u}^i}$, changed a few dummy indices, and used the fact that $g_{\alpha\beta} = g_{\beta\alpha}$. Similarly

$$\frac{\partial \bar{g}_{ki}}{\partial \bar{u}^j} = \frac{\partial g_{\gamma\alpha}}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k}\frac{\partial u^\alpha}{\partial \bar{u}^i} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^j\,\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^i} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^j\,\partial \bar{u}^i}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

and

$$\frac{\partial \bar{g}_{ij}}{\partial \bar{u}^k} = \frac{\partial g_{\gamma\alpha}}{\partial u^\beta}\frac{\partial u^\beta}{\partial \bar{u}^k}\frac{\partial u^\gamma}{\partial \bar{u}^i}\frac{\partial u^\alpha}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^k\,\partial \bar{u}^i}\frac{\partial u^\gamma}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^k\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^i}$$

It follows from equation *(10.25)*, page 214, that

$$\bar{\Gamma}_{ijk} = \frac{1}{2}\left[\frac{\partial \bar{g}_{jk}}{\partial \bar{u}^i} + \frac{\partial \bar{g}_{ki}}{\partial \bar{u}^j} - \frac{\partial \bar{g}_{ij}}{\partial \bar{u}^k}\right]$$
$$= \frac{1}{2}\left[\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right]\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$
$$= \left\{\Gamma_{\alpha\beta\gamma}\frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma}\frac{\partial^2 u^\alpha}{\partial \bar{u}^i\,\partial \bar{u}^j}\right\}\frac{\partial u^\gamma}{\partial \bar{u}^k}$$

10.28. Prove Theorem 10.2: Let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch on a surface of class $\geqq 2$ such that the coefficients of the Gauss-Weingarten equations are of class C^1. Then the mixed derivatives $\mathbf{x}_{uuv}, \mathbf{x}_{uvu}, \mathbf{x}_{vuv}, \mathbf{x}_{vvu}$ exist and satisfy equation (10.6) if and only if the first and second fundamental coefficients satisfy the compatibility equations (10.7) and (10.8).

Since it is assumed the coefficients of the Gauss equations $\mathbf{x}_{ij} = \Gamma^\alpha_{ij}\mathbf{x}_\alpha + b_{ij}\mathbf{N}$ are of class C^1, we can compute the third order derivatives.

$$\begin{aligned}\frac{\partial \mathbf{x}_{ij}}{\partial u^k} = \mathbf{x}_{ijk} &= (\Gamma^\alpha_{ij})_k\mathbf{x}_\alpha + \Gamma^\alpha_{ij}\mathbf{x}_{\alpha k} + (b_{ij})_k\mathbf{N} + b_{ij}\mathbf{N}_k \\ &= (\Gamma^\alpha_{ij})_k\mathbf{x}_\alpha + \Gamma^\alpha_{ij}[\Gamma^\beta_{\alpha k}\mathbf{x}_\beta + b_{\alpha k}\mathbf{N}] + (b_{ij})_k\mathbf{N} + b_{ij}(-b^\alpha_k\mathbf{x}_\alpha) \\ &= [(\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - b_{ij}b^\alpha_k]\mathbf{x}_\alpha + [\Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k]\mathbf{N}\end{aligned}$$

Observe that we used the Weingarten equation $\mathbf{N}_i = -b^\alpha_i\mathbf{x}_\alpha$. Similarly

$$\mathbf{x}_{ikj} = [(\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ik}b^\alpha_j]\mathbf{x}_\alpha + [\Gamma^\alpha_{ik}b_{\alpha j} + (b_{ik})_j]\mathbf{N}$$

Now the third order derivatives are independent of the order of differentiation if and only if $\mathbf{x}_{ijk} = \mathbf{x}_{ikj}$, $i, k, j = 1, 2$, or, if and only if

$$\begin{aligned}\mathbf{x}_{ijk} - \mathbf{x}_{ikj} &= [(\Gamma^\alpha_{ij})_k - (\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ij}b^\alpha_k + b_{ik}b^\alpha_j]\mathbf{x}_\alpha \\ &\quad + [\Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k - \Gamma^\alpha_{ik}b_{\alpha j} - (b_{ik})_j]\mathbf{N} = 0\end{aligned}$$

Since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{N}$ are independent, the equation is equivalent to

(a) $\quad (\Gamma^\alpha_{ij})_k - (\Gamma^\alpha_{ik})_j + \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k} - \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - b_{ij}b^\alpha_k + b_{ik}b^\alpha_j = 0, \qquad \alpha, i, j, k = 1, 2$

(b) $\quad \Gamma^\alpha_{ij}b_{\alpha k} + (b_{ij})_k - \Gamma^\alpha_{ik}b_{\alpha j} - (b_{ik})_j = 0, \qquad i, j, k = 1, 2$

We consider first equation (b). Observe that the equation is obviously satisfied if $j = k$. Also the left hand side simply changes sign if j and k are interchanged. Thus (b) is equivalent to the two equations obtained by taking $i = 1,\ j = 1,\ k = 2$ and $i = 2,\ j = 1,\ k = 2$:

$$(b_{11})_2 - (b_{12})_1 = \Gamma^\alpha_{12}b_{\alpha 1} - \Gamma^\alpha_{11}b_{\alpha 2}, \qquad (b_{21})_2 - (b_{22})_1 = \Gamma^\alpha_{22}b_{\alpha 1} - \Gamma^\alpha_{21}b_{\alpha 2}$$

If we expand the right hand sides of the above and use $b_{11} = L,\ b_{12} = b_{21} = M,\ b_{22} = N,\ u = u^1$ and $v = u^2$ we obtain the Mainardi-Codazzi equations (10.7):

$$\begin{aligned}L_v - M_u &= \Gamma^1_{12}L + (\Gamma^2_{12} - \Gamma^1_{11})M - \Gamma^2_{11}N \\ M_v - N_u &= \Gamma^1_{22}L + (\Gamma^2_{22} - \Gamma^1_{12})M - \Gamma^2_{12}N\end{aligned}$$

We now consider equation (a) above which we can write, using equation (10.32), as

(c) $\quad R^\alpha_{ijk} = (\Gamma^\alpha_{ik})_j - (\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}, \qquad \alpha, i, j, k = 1, 2$

From equation (10.31) and Problem 10.26, the above is equivalent to

$$R_{pijk} = g_{\alpha p}R^\alpha_{ijk} = g_{\alpha p}(\Gamma^\alpha_{ik})_j - g_{\alpha p}(\Gamma^\alpha_{ij})_k + g_{\alpha p}\Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - g_{\alpha p}\Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

From the skew-symmetric properties of R_{pijk} (see equation (10.33)) and equations (10.34) and (10.35), it follows that the above equation is equivalent to the single equation,

$$R_{1212} = g_{\alpha 1}(\Gamma^\alpha_{22})_1 - g_{\alpha 1}(\Gamma^\alpha_{21})_2 + g_{\alpha 1}\Gamma^\beta_{22}\Gamma^\alpha_{\beta 1} - g_{\alpha 1}\Gamma^\beta_{21}\Gamma^\alpha_{\beta 2}$$

or expanding and collecting terms,

$$\begin{aligned}R_{1212} &= g_{11}\{(\Gamma^1_{22})_1 - (\Gamma^1_{21})_2 + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22}\} \\ &\quad + g_{21}\{(\Gamma^2_{22})_1 - (\Gamma^2_{21})_2 + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{12}\}\end{aligned}$$

If we use $g_{11} = E,\ g_{21} = F,\ u^1 = u,\ u^2 = v,\ \Gamma^i_{12} = \Gamma^i_{21}$, and, from equation (10.34), $R_{1212} = LM - N^2$, we obtain

$$\begin{aligned}LM - N^2 &= E\{(\Gamma^1_{22})_u - (\Gamma^1_{12})_v + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}\} \\ &\quad + F\{(\Gamma^2_{22})_u - (\Gamma^2_{12})_v + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12}\}\end{aligned}$$

which is the third of the compatibility equations (10.8).

10.29. Prove Theorem 10.9: $R_{mijk} = (\Gamma_{ikm})_j - (\Gamma_{ijm})_k + \Gamma^\alpha_{ij}\Gamma_{mk\alpha} - \Gamma^\alpha_{ik}\Gamma_{mj\alpha}$.

From Problem 10.28, equation (c),

$$R^\alpha_{ijk} = (\Gamma^\alpha_{ik})_j - (\Gamma^\alpha_{ij})_k + \Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - \Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

From Problem 10.26,

$$R_{mijk} = g_{\alpha m}R^\alpha_{ijk} = g_{\alpha m}(\Gamma^\alpha_{ik})_j - g_{\alpha m}(\Gamma^\alpha_{ij})_k + g_{\alpha m}\Gamma^\beta_{ik}\Gamma^\alpha_{\beta j} - g_{\alpha m}\Gamma^\beta_{ij}\Gamma^\alpha_{\beta k}$$

Now $$g_{\alpha m}(\Gamma^{\alpha}_{ik})_j = (g_{\alpha m}\Gamma^{\alpha}_{ik})_j - (g_{\alpha m})_j\Gamma^{\alpha}_{ik} = (\Gamma_{ikm})_j - (\Gamma_{\alpha jm} + \Gamma_{mj\alpha})\Gamma^{\alpha}_{ik}$$

where we used equation (*10.24*) and Problem 10.23. Similarly

$$g_{\alpha m}(\Gamma^{\alpha}_{ij})_k = (\Gamma_{ijm})_k - (\Gamma_{\alpha km} + \Gamma_{mk\alpha})\Gamma^{\alpha}_{ij}$$

Also $g_{\alpha m}\Gamma^{\beta}_{ik}\Gamma^{\alpha}_{\beta j} = \Gamma^{\beta}_{ik}g_{\alpha m}\Gamma^{\alpha}_{\beta j} = \Gamma^{\beta}_{ik}\Gamma_{\beta jm}$ and similarly $g_{\alpha m}\Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta k} = \Gamma^{\beta}_{ij}\Gamma_{\beta km}$. Thus substituting in the above

$$R_{mijk} = (\Gamma_{ikm})_j - \Gamma^{\alpha}_{ik}\Gamma_{\alpha jm} - \Gamma^{\alpha}_{ik}\Gamma_{mj\alpha} - (\Gamma_{ijm})_k + \Gamma^{\alpha}_{ij}\Gamma_{\alpha km} + \Gamma^{\alpha}_{ij}\Gamma_{mk\alpha} + \Gamma^{\beta}_{ik}\Gamma_{\beta jm} - \Gamma^{\beta}_{ij}\Gamma_{\beta km}$$

which gives the required result.

Supplementary Problems

THEORY OF SURFACES

10.30. Obtain the Christoffel symbols Γ^k_{ij} for the cylinder $\mathbf{x} = \mathbf{y}(u) + v\mathbf{g}$, $\mathbf{g}$ = constant, $|\mathbf{g}| = 1$.
Ans. $\Gamma^1_{11} = \mathbf{y}'\cdot\mathbf{y}''/|\mathbf{y}'\times\mathbf{g}|^2$, $\Gamma^2_{11} = (g\cdot y')(y'\cdot y'')/|y'\times g|^2$, otherwise $\Gamma^k_{ij} = 0$.

10.31. Verify that the functions $E = 1 + 4u^2$, $F = -4uv$, $G = 1 + 4v^2$, $L = 2(4u^2+4v^2+1)^{-1/2}$, $M = 0$, $N = -2(4u^2+4v^2+1)^{-1/2}$ satisfy the compatibility conditions, equations (*10.7*) and (*10.8*), page 203.

10.32. Using the Weingarten equations, prove that $\mathbf{N}_u \times \mathbf{N}_v = (EG - F^2)K\mathbf{N}$.

10.33. Solve the Gauss-Weingarten equations for the surface whose fundamental coefficients are $E = 1$, $F = 0$, $G = 1$, $L = -1$, $M = 0$, $N = 0$. *Ans.* Circular cylinder of radius 1.

10.34. Derive Rodrigues' formula from the Weingarten equations.

10.35. If the parameter curves on a patch are orthogonal, prove that

$$K = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right]$$

10.36. Prove that $f(P) = \mathbf{x}\cdot\mathbf{x}$ is a continuous function of a point P on a surface.

10.37. Prove that the principal curvatures $\kappa_1(P)$ and $\kappa_2(P)$ are continuous functions of a point P on an oriented surface.

10.38. Prove Theorem 10.6: The only connected and closed surfaces of class ≥ 2 of which all points are planar points are planes.

10.39. Prove that the spheres are the only connected compact surfaces with positive Gaussian curvature and constant mean curvature.

TENSORS

10.40. If A_i and B_j are the components of two covariant vectors, show that the outer product $C_{ij} = A_iB_j$ are the components of a covariant tensor of rank 2.

10.41. Show that $\det\begin{pmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{pmatrix} = e^{ijk}a^1_i a^2_j a^3_k$ where e^{ijk} is defined in Example 10.6, page 210.

10.42. Show that $\delta^{\alpha\beta}_{ij}A_{\alpha\beta} = A_{ij} - A_{ji}$.

10.43. If A^{ij} are the components of an absolute contravariant tensor and $A_{i\alpha}A^{\alpha j} = \delta_i^j$, show that $A^{\alpha j}$ are the components of an absolute covariant tensor. The two tensors are said to be *reciprocal.*

10.44. If A^{ij} and A_{ij} are the components of reciprocal symmetric tensors, and if x_i are the components of a covariant vector, show that $A_{ij}x^ix^j = A^{ij}x_ix_j$ where $x_i = A^{i\alpha}x_\alpha$.

10.45. Show that the quantities $e_{ijk} = e^{ijk}$, where the e^{ijk} are defined in Example 10.6(*f*), page 211, are the components of a covariant tensor of rank 3 and weight -1.

10.46. Let $\epsilon_{11} = 0$, $\epsilon_{12} = \sqrt{g}$, $\epsilon_{21} = -\sqrt{g}$, $\epsilon_{22} = 0$, where $g = g_{11}g_{22} - (g_{12})^2$. Show that the ϵ_{ij}, $i, j = 1, 2$, are the components of a skew-symmetric covariant tensor such that $\bar{\epsilon}_{11} = 0$, $\bar{\epsilon}_{12} = \sqrt{\bar{g}}$, $\epsilon_{21} = -\sqrt{\bar{g}}$, $\epsilon_{22} = 0$.

10.47. Let $\epsilon^{ij} = \epsilon_{\alpha\beta}\, g^{i\alpha}g^{j\beta}$ where $\epsilon_{\alpha\beta}$ is defined in the preceding problem. Show that $\epsilon^{11} = 0$, $\epsilon^{12} = 1/\sqrt{g}$, $\epsilon^{21} = -1/\sqrt{g}$, $\epsilon^{22} = 0$.

10.48. Show that $b_i^\beta\, b_{\beta j} - b_j^\beta\, b_{\beta i} = 0$, $i, j = 1, 2$.

10.49. Prove that $$\bar{\Gamma}_{ij}^k = \left[\Gamma_{\alpha\beta}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^i}\frac{\partial u^\beta}{\partial \bar{u}^j} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^i\,\partial \bar{u}^j}\right]\frac{\partial \bar{u}^k}{\partial u^\gamma}.$$

10.50. Prove that $$\frac{\partial g^{ij}}{\partial u^k} = -g^{\alpha j}\Gamma_{\alpha k}^i - g^{\alpha i}\Gamma_{\alpha k}^j.$$

10.51. Show that

$$R_{112}^1 = R_{221}^2 = -R_{121}^1 = -R_{212}^2 = F\frac{LN - M^2}{EG - F^2}$$

$$R_{212}^1 = -R_{221}^1 = G\frac{LN - M^2}{EG - F^2}$$

$$R_{121}^2 = -R_{121}^2 = E\frac{LN - M^2}{EG - F^2}$$

and otherwise $R_{ijk}^p = 0$.

10.52. Prove that $$R_{ijk}^p = \frac{\partial \Gamma_{ik}^p}{\partial u^j} - \frac{\partial \Gamma_{ij}^p}{\partial u^k} + \Gamma_{ik}^\alpha \Gamma_{\alpha j}^p - \Gamma_{ij}^\alpha \Gamma_{\alpha k}^p.$$

10.53. Prove that $$\frac{1}{2}\frac{\partial \log g}{\partial u^1} = \Gamma_{11}^1 + \Gamma_{12}^2, \quad \frac{1}{2}\frac{\partial \log g}{\partial u^2} = \Gamma_{12}^1 + \Gamma_{22}^2.$$