Solved Problems

MAPPINGS OF SURFACES

11.1. **Prove that a regular differentiable mapping f of a surface S into a surface S^* is a continuous mapping of S into S^*.**

Suppose $\mathbf{f}(P_0) = P_0^*$ is a point on S^* and $S_\epsilon(P_0^*)$ is an arbitrary neighborhood of P_0^*. From the corollary on page 229, there exists a patch D on S containing P_0 on which f is continuous. Thus there exists a neighborhood $S_{\delta_1}(P_0)$ such that $\mathbf{f}(P) \in S_\epsilon(P_0^*)$ for P in $S_{\delta_1}(P_0) \cap D$. Now, from Problem 8.13, for any point P_0 on a patch D on a surface S there exists a neighborhood $S_{\delta_2}(P_0)$ such that $S_{\delta_2}(P_0) \cap S \subset D$. Thus for P in $S_\delta(P_0) \cap S$, where $\delta = \min(\delta_1, \delta_2)$, we have P in $S_{\delta_1}(P_0) \cap D$ and hence $\mathbf{f}(P)$ in $S_\epsilon(P_0^*)$. It follows that f is continuous at P_0. Since P_0 is arbitrary, f is a continuous mapping of S into S^*.

11.2. **Prove that if f is a regular differentiable mapping of S into S^* and g is a regular differentiable mapping of S^* into S^{**}, then the composite mapping $\mathbf{g} \circ \mathbf{f}$ is a regular differentiable mapping of S into S^{**}.**

Let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch on S defined on U. It is required to show that $\mathbf{x}^{**} = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u, v))$ is a regular parametric representation on S^{**}. Since f is a regular differentiable mapping of S into S^*, it follows from Theorem 11.1 that there exists a neighborhood $S(u, v)$ of each (u, v) in U such that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a coordinate patch on S^* for (u, v) in $S(u, v)$. Since g is a regular differentiable mapping of S^* into S^{**}, then $\mathbf{x}^{**} = \mathbf{g}(\mathbf{x}^*(u, v)) = \mathbf{g}(\mathbf{f}(\mathbf{x}(u, v))) = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u, v))$ is a regular parametric representation on S^{**} for (u, v) in $S(u, v)$. Since (u, v) is an arbitrary point in U, $\mathbf{x}^{**} = (\mathbf{g} \circ \mathbf{f})(\mathbf{x}(u, v))$ is a regular parametric representation on S^{**} defined on U, which is the required result.

11.3. **Let f be a mapping of S into S^* such that for every coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ of a basis of patches for S the mapping $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a regular parametric representation of class C^r. Prove that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a regular parametric representation of class C^r for all coordinate patches $\mathbf{x} = \mathbf{x}(u, v)$ on S and hence f is a regular differentiable mapping of class C^r of S into S^*.**

Let $\mathbf{x} = \mathbf{x}(u, v)$ be an arbitrary patch on S defined on U. It is required to show that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a regular parametric representation on S^*. Let (u, v) be an arbitrary point in U and let P be the point on $\mathbf{x} = \mathbf{x}(u, v)$ corresponding to (u, v). Let $\mathbf{x} = \mathbf{y}(\theta, \phi)$ be a patch of the basis which contains P. From Theorem 8.3, page 157, the intersection of $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{y}(\theta, \phi)$ is a patch on S defined on an open set W containing (u, v) on which $\theta = \theta(u, v)$, $\phi = \phi(u, v)$ is an allowable parameter transformation. Since $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v)) = \mathbf{f}(\mathbf{y}(\theta(u, v), \phi(u, v)))$, where $\mathbf{f}(\mathbf{y}(\theta, \phi))$ is a regular parametric representation, it follows that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a regular parametric representation defined on W. Since (u, v) is an arbitrary point in U, it follows that $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u, v))$ is a regular parametric representation defined on U.

11.4. **If f is a 1-1 regular differentiable mapping of a surface S onto a surface S^*, prove that $\mathbf{f}^{-1}$ is a regular differentiable mapping of S^* onto S.**

Let $\mathbf{x}^* = \mathbf{x}^*(u, v)$ be a patch on S^* defined on U. It is required to show that $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u, v))$ is a regular parametric representation on S. As in the preceding problems, it is sufficient to show that $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u, v))$ is a regular parametric representation for some neighborhood of an arbitrary point (u, v) in U. We let P^* denote the image of (u, v) under $\mathbf{x}^* = \mathbf{x}^*(u, v)$ and P the image of P^* under $\mathbf{f}^{-1}$. Now let $\mathbf{x} = \mathbf{x}(\theta, \phi)$ be a patch on S containing P. Since f is a regular differentiable mapping of S into S^*, $\mathbf{x}^* = \mathbf{x}^*(\theta, \phi) = \mathbf{f}(\mathbf{x}(\theta, \phi))$ is a regular parametric representation on S^* which contains P^*. From Theorem 11.1, page 228, we can assume that $\mathbf{x}^* = \mathbf{x}^*(\theta, \phi)$ is a patch. From Theorem 8.3, page 157, the intersection of the patches $\mathbf{x}^* = \mathbf{x}^*(u, v)$ and $\mathbf{x}^* = \mathbf{x}^*(\theta, \phi)$ on S^* is a patch which contains P^* such that $\theta = \theta(u, v), \phi = \phi(u, v)$ is an allowable parameter transformation. But then on the intersection, $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{x}^*(u, v)) = \mathbf{x}(\theta(u, v), \phi(u, v))$ is a regular parametric representation on S, which completes the proof.

ISOMETRIC MAPPINGS

11.5. If f is an isometry of a surface S onto a surface S^*, prove that $\mathbf{f}^{-1}$ is an isometry of S^* onto S.

From Problem 11.4, $\mathbf{f}^{-1}$ is a regular differential mapping of S^* on S. Thus for any arc C^* on S^*, $\mathbf{f}^{-1}(C^*)$ is a regular arc on S and, since f is an isometric mapping of S into S^*,

$$L(\mathbf{f}^{-1}(C)) = L(\mathbf{f}(\mathbf{f}^{-1}(C^*)) = L(C^*)$$

where $L(C^*)$ is the length of C^*, which proves that $\mathbf{f}^{-1}$ is an isometry from S^* onto S.

11.6. Prove Theorem 11.4: The intrinsic distance $D(P,Q)$ from a point P to a point Q on a surface satisfies

(i) $D(P,Q) = D(Q,P)$

(ii) $D(P,R) \leq D(P,Q) + D(Q,R)$

(iii) $D(P,Q) \geq 0$, $D(P,Q) = 0$ iff $P = Q$.

(i) Since the length $L(C)$ of a regular arc C from P to Q is independent of the orientation of C, the set of numbers $L(C)$ for all regular arcs C from P to Q is independent of the orientation of C. Thus $D(P,Q)$, which is the infimum of the numbers $L(C)$, is independent of the orientation of C. Hence $D(P,Q) = D(Q,P)$.

(ii) Since $D(P,Q)$ is the infimum of the lengths of the arcs from P to Q, for arbitrary $\epsilon > 0$ there exists a regular arc C_1 from P to Q such that $L(C_1) \leq D(P,Q) + \epsilon$. For the same reason there exists a regular arc C_2 from Q to R such that $L(C_2) \leq D(Q,R) + \epsilon$. Now the arc obtained by joining C_2 to C_1 will in general have a "corner" at Q and hence not be a regular arc from P to R. However, it can be shown that there exists a regular arc from P to R which at worst is slightly longer. Namely, there exists an arc C from P to R such that $L(C) \leq L(C_1) + L(C_2) + \epsilon$. But then it follows that

$$D(P,R) \leq L(C) \leq L(C_1) + L(C_2) + \epsilon \leq D(P,Q) + D(Q,R) + 3\epsilon$$

Since ϵ is arbitrary, we have $D(P,R) \leq D(P,Q) + D(Q,R)$.

(iii) Since $L(C) \geq 0$, for any arc C from P to Q, $D(P,Q) \geq 0$. If $P = Q$, for any $\epsilon > 0$, there exists a regular arc C from P to Q such that $L(C) \leq \epsilon$. Since $D(P,Q) \leq L(C) \leq \epsilon$ for arbitrary ϵ, it follows that $D(P,Q) = 0$. Conversely, suppose $D(P,Q) = 0$. Then for an arbitrary $\epsilon > 0$ there exists an arc C from P to Q such that $L(C) \leq D(P,Q) + \epsilon = \epsilon$. But the Euclidean distance $|P - Q| \leq L(C)$. Since ϵ is arbitrary, $|P - Q| = 0$ or $P = Q$.

11.7. Let f be a local isometric mapping of a surface S into a surface S^*. Prove that for any two points P and Q on S, the intrinsic distance $D(P,Q) = D(\mathbf{f}(P), \mathbf{f}(Q))$.

Since $D(P,Q)$ is the infimum of the lengths of the arcs between P and Q, given an arbitrary $\epsilon > 0$, there exists an arc C joining P and Q such that its length $L(C)$ satisfies $L(C) \leq D(P,Q) + \epsilon$. Now let $C^* = \mathbf{f}(C)$. Since f is a local isometry, $L(C^*) = L(C)$. Thus $D(\mathbf{f}(P), \mathbf{f}(Q)) \leq L(C^*) = L(C) \leq D(P,Q) + \epsilon$. Since ϵ is arbitrary, the required result follows.

11.8. Let $\mathbf{y} = \mathbf{y}(s,v) = \mathbf{x}(s) + v\mathbf{t}(s)$, $v > 0$, be a branch of the tangent surface of a natural represented curve $\mathbf{x} = \mathbf{x}(s)$ without points of inflection (see Problem 8.19, page 167). Prove that a neighborhood of every point on the tangent surface can be mapped isometrically onto a subset of the plane.

From the fundamental theorem for curves, there exists a natural representation of a curve $\mathbf{x}^* = \mathbf{x}^*(s)$ in the x_1x_2 plane such that along $\mathbf{x}^* = \mathbf{x}^*(s)$ the curvature $\kappa^*(s)$ is equal to the curvature $\kappa(s)$ along $\mathbf{x} = \mathbf{x}(s)$. Given a point P on the tangent surface and a patch $\mathbf{y} = \mathbf{y}(s,v)$ containing P, define the mapping f of the patch into the plane by $\mathbf{y}^* = \mathbf{f}(\mathbf{y}(s,v)) = \mathbf{f}(\mathbf{x}(s) + v\mathbf{t}(s)) = \mathbf{x}^*(s) + v\mathbf{t}^*(s)$. Here $\mathbf{y}_s^* = \dot{\mathbf{x}}^* + v\dot{\mathbf{t}}^* = \mathbf{t}^* + v\kappa^*\mathbf{n}^*$ and $\mathbf{y}_v^* = \mathbf{t}^*$ are continuous and $|\mathbf{y}_s^* \times \mathbf{y}_v^*| = v\kappa^* \neq 0$, since $v > 0$ and $\kappa^* = \kappa \neq 0$; hence f is a regular differentiable mapping. From Theorem 11.1, page 228, we can assume that the patch containing P is small enough so that f is 1-1. Now along the patch $\mathbf{y} = \mathbf{y}(s,v)$ we have

$$E = \mathbf{y}_s \cdot \mathbf{y}_s = (\mathbf{t} + v\kappa\mathbf{n}) \cdot (\mathbf{t} + v\kappa\mathbf{n}) = 1 + v^2\kappa^2$$

$$F = \mathbf{y}_s \cdot \mathbf{y}_v = (\mathbf{t} + v\kappa\mathbf{n}) \cdot (\mathbf{t}) = 1, \qquad G = \mathbf{y}_v \cdot \mathbf{y}_v = \mathbf{t} \cdot \mathbf{t} = 1$$

On $\mathbf{y}^* = \mathbf{f}(\mathbf{y}(s,v)) = \mathbf{x}^*(s) + v\mathbf{t}^*(s)$ we have

$$E^* = \mathbf{y}_s^* \cdot \mathbf{y}_s^* = 1 + v^2(\kappa^*)^2, \qquad F^* = \mathbf{y}_s^* \cdot \mathbf{y}_v^* = 1, \qquad G^* = \mathbf{y}_v^* \cdot \mathbf{y}_v^* = 1$$

But $\kappa = \kappa^*$. Hence $E = E^*$, $F = F^*$, $G = G^*$. From Theorem 11.3, page 230, it follows that $\mathbf{f}$ is an isometry.

11.9. If $\mathbf{f}$ is an isometry from S onto S^* and $\mathbf{x} = \mathbf{x}(u,v)$ is a patch on S, prove that $E = E^*$, $F = F^*$, $G = G^*$, where E, F and G are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and E^*, F^* and G^* are the first fundamental coefficients on $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$.

Let (u,v) be an arbitrary point in the domain of $\mathbf{x} = \mathbf{x}(u,v)$. Let $u = u(t)$, $v = v(t)$, $a \leq t \leq b$, be an arbitrary arc through (u,v) and let C_τ and C_τ^* be the images on S and S^* respectively of $u = u(t)$, $v = (t)$ on the interval $a \leq t \leq \tau$. Since $\mathbf{f}$ is an isometry of S onto S^*,

$$L(C_\tau) = \int_a^\tau \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt = L(C_\tau^*)$$

$$= \int_a^\tau \sqrt{E^*\left(\frac{du}{dt}\right)^2 + 2F^*\frac{du}{dt}\frac{dv}{dt} + G^*\left(\frac{dv}{dt}\right)^2}\, dt$$

But the above is valid for all τ. Hence for all t, and in particular at (u,v),

$$E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 = E^*\left(\frac{du}{dt}\right)^2 + 2F^*\frac{du}{dt}\frac{dv}{dt} + G^*\left(\frac{dv}{dt}\right)^2$$

But the curve $u = u(t)$, $v = v(t)$ through (u,v) is arbitrary. Hence the above equation is also valid at (u,v) for all du/dt, dv/dt. It follows that $E = E^*$, $F = F^*$ and $G = G^*$ at (u,v). Since (u,v) is arbitrary, the result follows.

11.10. A regular differentiable mapping $\mathbf{f}$ of a surface S into a surface S^* is said to be *conformal* if for every patch $\mathbf{x} = \mathbf{x}(u,v)$ on S there exists a function $\lambda(u,v) > 0$ such that for all (u,v)

$$E = \lambda E^*, \quad F = \lambda F^* \quad \text{and} \quad G = \lambda G^*$$

where E, F and G and E^*, F^* and G^* are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$ respectively. Prove that a conformal mapping preserves the angle between intersecting oriented curves. By the angle between two intersecting oriented curves $\mathbf{x} = \mathbf{x}(t)$ and $\boldsymbol{\xi} = \boldsymbol{\xi}(\tau)$ we mean the angle $\theta = \measuredangle(\mathbf{x}', \boldsymbol{\xi}')$, between their tangents at the point of intersection.

Suppose $\mathbf{x} = \mathbf{x}(u,v)$ is a patch containing P and $\mathbf{x} = \mathbf{x}(u(t), v(t))$ and $\boldsymbol{\xi} = \mathbf{x}(\eta(\tau), \zeta(\tau))$ are two curves on S intersecting at P with tangents $\mathbf{x}' = \mathbf{x}_u u' + \mathbf{x}_v v'$ and $\boldsymbol{\xi}' = \mathbf{x}_u \eta' + \mathbf{x}_v \zeta'$ at P respectively. If $\theta = \measuredangle(\mathbf{x}', \boldsymbol{\xi}')$, then from equation (9.6), page 172,

$$\cos\theta = \frac{Eu'\eta' + F(u'\zeta' + v'\eta') + Gv'\zeta'}{[E(u')^2 + 2Fu'v' + G(v')^2]^{1/2}\,[E(\eta')^2 + 2F\eta'\zeta' + G(\zeta')^2]^{1/2}}$$

On the other hand if θ^* is the angle between the tangents $\mathbf{x}^{*\prime} = \mathbf{x}_u^* u' + \mathbf{x}_v^* v'$ and $\boldsymbol{\xi}^{*\prime} = \mathbf{x}_u^* \eta' + \mathbf{x}_v^* \zeta'$ of the images $\mathbf{x}^* = \mathbf{x}^*(u(t), v(t))$ and $\boldsymbol{\xi}^* = \mathbf{x}^*(\eta(\tau), \zeta(\tau))$ of the curves on S^*, then

$$\cos\theta^* = \frac{E^*u'\eta' + F^*(u'\zeta' + v'\eta') + G^*v'\zeta'}{[E^*(u')^2 + 2F^*u'v' + G^*(v')^2]^{1/2}\,[E^*(\eta')^2 + 2F^*\eta'\zeta' + G^*(\zeta')^2]^{1/2}}$$

But at corresponding points $E = \lambda E^*$, $F = \lambda F^*$ and $G = \lambda G^*$. Hence $\cos\theta = \cos\theta^*$ or $\theta = \theta^*$, which completes the proof.

11.11. Two surfaces S and S^* are said to be *applicable* if there exists a continuous family $\mathbf{f}_\lambda$, $0 \leq \lambda \leq 1$, of mappings of S into E^3 such that (i) $\mathbf{f}_0(S) = S$, (ii) $\mathbf{f}_1(S) = S^*$, (iii) for all λ the mappings $\mathbf{f}_\lambda$ are isometric mappings of S onto $\mathbf{f}_\lambda(S)$. Intuitively S and S^* are applicable if S can be continuously and isometrically bent onto S^*. If S and S^* are applicable, we say that S^* can be obtained from S by *bending*. A property of a surface which is invariant under such a continuous family of isometries is called a *bending invariant*. Clearly if S and S^* are applicable, then they are isometric. The converse is not true in general. Prove that a neighborhood of every point on a branch of the tangent surface of a curve can be bent onto the plane (see Problem 11.8).

Let $\mathbf{y} = \mathbf{y}(s,v) = \mathbf{x}(s) + v\mathbf{t}(s)$, $v > 0$, be the tangent surface of a curve $\mathbf{x} = \mathbf{x}(s)$ without points of inflection. From the fundamental theorem of curves for each λ, $0 \leq \lambda \leq 1$, there exists a curve $\mathbf{x} = \mathbf{x}_\lambda(s)$, with curvature $\kappa(s)$ and torsion $(1-\lambda)\tau(s)$, where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion along $\mathbf{x} = \mathbf{x}(s)$. Note that $\mathbf{x} = \mathbf{x}_0(s) = \mathbf{x}(s)$ and that $\mathbf{x} = \mathbf{x}_1(s)$ is a plane curve since its torsion is zero. It can also be shown that $\mathbf{x}_\lambda(s)$ is continuous in λ. Now consider the family of mappings $\mathbf{f}_\lambda$ of the tangent surface defined by $\mathbf{x} = \mathbf{f}_\lambda(\mathbf{y}(s,v)) = \mathbf{x}_\lambda(s) + v\mathbf{t}_\lambda(s)$, $v > 0$. As in Problem 11.8 it is easily verified that for each λ, $\mathbf{f}_\lambda$ is a regular differentiable mapping of a patch on the tangent surface onto its image. Clearly $\mathbf{f}_0(\mathbf{y}(s,v)) = \mathbf{y}(s,v)$ and $\mathbf{f}_1(\mathbf{y}(s,v))$ is a subset of the plane generated by the tangents to the plane curve $\mathbf{x} = \mathbf{x}_1(s)$. Finally, for each λ, $E_\lambda = (\mathbf{x}_\lambda)_s \cdot (\mathbf{x}_\lambda)_s = 1 + v^2\kappa^2 = E_0$, $F_\lambda = (\mathbf{x}_\lambda)_s \cdot (\mathbf{x}_\lambda)_v = 1 = F$, and $G_\lambda = (\mathbf{x}_\lambda)_v \cdot (\mathbf{x}_\lambda)_v = 1 = E$, which proves the proposition.

GEODESICS

11.12. Determine the geodesics on the right circular cone

$\mathbf{x} = (u \sin\alpha \cos\theta)\mathbf{e}_1 + (u \sin\alpha \sin\theta)\mathbf{e}_2 + (u\cos\alpha)\mathbf{e}_3 \qquad \alpha = \text{constant},\ 0 < \alpha < \pi/2,\ u > 0$

by solving equations (*11.7*), page 234.

$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_\theta = 0$, $G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = u^2 \sin^2\alpha$; $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^2_{11} = \Gamma^2_{22} = 0$, $\Gamma^1_{22} = -u\sin^2\alpha$, $\Gamma^2_{12} = 1/u$. Thus the second of equations (*11.7*) is $\dfrac{d^2\theta}{ds^2} = -(2/u)\dfrac{du}{ds}\dfrac{d\theta}{ds}$. Setting $\phi = \dfrac{d\theta}{ds}$ gives $\dfrac{1}{\phi}\dfrac{d\phi}{ds} = \dfrac{-2}{u}\dfrac{du}{ds}$. Hence $\log\phi = -2\log u + K$ or $\phi = \dfrac{d\theta}{ds} = C/u^2\sin^2\alpha$, where $C = e^K \sin^2\alpha$. Since s is arc length,

$$1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_\theta\frac{d\theta}{ds}\right|^2 = E\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{d\theta}{ds} + G\left(\frac{d\theta}{ds}\right)^2$$

or $1 = \left(\dfrac{du}{ds}\right)^2 + u^2\sin^2\alpha\left(\dfrac{d\theta}{ds}\right)^2$. Substituting $d\theta/ds = C/u^2\sin^2\alpha$ gives

$$du/ds = \sqrt{u^2\sin^2\alpha - C^2}/u\sin\alpha$$

Hence $\qquad du/d\theta = (1/C)u\sin\alpha\sqrt{u^2\sin^2\alpha - C^2} \qquad$ or $\qquad u = A\sec[(\sin\alpha)\theta + B]$

where A = constant, B = constant.

11.13. Show that the geodesic curvature κ_g of a naturally represented curve $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ of class C^2 on a patch $\mathbf{x} = \mathbf{x}(u,v)$ of class C^2 is given by

$$\kappa_g = \left[\Gamma^2_{11}\left(\frac{du}{ds}\right)^3 + (2\Gamma^2_{12} - \Gamma^1_{11})\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right) + (\Gamma^2_{22} - 2\Gamma^1_{12})\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2 - \Gamma^1_{22}\left(\frac{dv}{ds}\right)^3 + \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right]\sqrt{EG-F^2}$$

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}, \quad \mathbf{k} = \frac{d\mathbf{t}}{ds} = \mathbf{x}_{uu}\left(\frac{du}{ds}\right)^2 + 2\mathbf{x}_{uv}\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + \mathbf{x}_{vv}\left(\frac{dv}{ds}\right)^2 + \mathbf{x}_u\frac{d^2u}{ds^2} + \mathbf{x}_v\frac{dv^2}{ds^2}$$

Thus from equation (*11.3*), page 233,

$$\kappa_g = [\mathbf{tkN}] = [\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}]\left(\frac{du}{ds}\right)^3 + (2[\mathbf{x}_u\mathbf{x}_{uv}\mathbf{N}] + [\mathbf{x}_v\mathbf{x}_{uu}\mathbf{N}])\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right)$$
$$+([\mathbf{x}_u\mathbf{x}_{vv}\mathbf{N}] + 2[\mathbf{x}_v\mathbf{x}_{uv}\mathbf{N}])\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)^2 + [\mathbf{x}_v\mathbf{x}_{vv}\mathbf{N}]\left(\frac{dv}{ds}\right)^3 + [\mathbf{x}_u\mathbf{x}_v\mathbf{N}]\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right)$$

Now from the Gauss equation $\mathbf{x}_{uu} = \Gamma^1_{11}\mathbf{x}_u + \Gamma^2_{11}\mathbf{x}_v + L\mathbf{N}$ on page 202, we obtain

$$[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}] = \Gamma^1_{11}[\mathbf{x}_u\mathbf{x}_u\mathbf{N}] + \Gamma^2_{11}[\mathbf{x}_u\mathbf{x}_v\mathbf{N}] + L[\mathbf{x}_u\mathbf{N}\mathbf{N}] = \Gamma^2_{11}[\mathbf{x}_u\mathbf{x}_v\mathbf{N}]$$

But $[\mathbf{x}_u\mathbf{x}_v\mathbf{N}] = (\mathbf{x}_u \times \mathbf{x}_v)\cdot(\mathbf{x}_u \times \mathbf{x}_v)/|\mathbf{x}_u \times \mathbf{x}_v| = |\mathbf{x}_u \times \mathbf{x}_v| = \sqrt{EG-F^2}$. Hence $[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}] = \Gamma^2_{11}\sqrt{EG-F^2}$. Similarly, $[\mathbf{x}_v\mathbf{x}_{uu}\mathbf{N}] = -\Gamma^1_{11}\sqrt{EG-F^2}$, $[\mathbf{x}_u\mathbf{x}_{uv}\mathbf{N}] = \Gamma^2_{12}\sqrt{EG-F^2}$, $[\mathbf{x}_v\mathbf{x}_{uv}\mathbf{N}] = -\Gamma^1_{12}\sqrt{EG-F^2}$, $[\mathbf{x}_u\mathbf{x}_{vv}\mathbf{N}] = \Gamma^2_{22}\sqrt{EG-F^2}$, $[\mathbf{x}_v\mathbf{x}_{vv}\mathbf{N}] = -\Gamma^1_{22}\sqrt{EG-F^2}$. Substituting in the equation for κ_g above gives the required result.

11.14. Prove Theorem 11.8: A natural representation of a curve $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ of class C^2 on a patch $\mathbf{x} = \mathbf{x}(u, v)$ of class C^2 is a geodesic if and only if $u(s)$ and $v(s)$ satisfy

$$\frac{d^2u}{ds^2} + \Gamma^1_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^1_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^1_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

$$\frac{d^2v}{ds^2} + \Gamma^2_{11}\left(\frac{du}{ds}\right)^2 + 2\Gamma^2_{12}\frac{du}{ds}\frac{dv}{ds} + \Gamma^2_{22}\left(\frac{dv}{ds}\right)^2 = 0$$

Recall that $\kappa_g = \mathbf{k}\cdot\mathbf{U}$ where $\mathbf{t}, \mathbf{U}, \mathbf{N}$ is an orthonormal triad. Thus $\mathbf{x} = \mathbf{x}(s)$ is a geodesic iff $\mathbf{k}\cdot\mathbf{U} = 0$. Since $\mathbf{k}$ is always orthogonal to $\mathbf{t}$, it follows that $\mathbf{x} = \mathbf{x}(s)$ is a geodesic if and only if $\mathbf{k}\cdot\mathbf{x}_u = 0$ and $\mathbf{k}\cdot\mathbf{x}_v = 0$. From $\mathbf{t} = \mathbf{x}_u\cdot(du/ds) + \mathbf{x}_v\cdot(dv/ds)$ we obtain

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \mathbf{x}_{uu}\left(\frac{du}{ds}\right)^2 + 2\mathbf{x}_{uv}\frac{du}{ds}\frac{dv}{ds} + \mathbf{x}_{vv}\left(\frac{dv}{ds}\right)^2 + \mathbf{x}_u\frac{d^2u}{ds^2} + \mathbf{x}_v\frac{d^2v}{ds^2}$$

Hence $\mathbf{x} = \mathbf{x}(s)$ is a geodesic if and only if

$$\mathbf{k}\cdot\mathbf{x}_u = (\mathbf{x}_{uu}\cdot\mathbf{x}_u)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_{uv}\cdot\mathbf{x}_u)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_{vv}\cdot\mathbf{x}_u)\left(\frac{dv}{ds}\right)^2 + (\mathbf{x}_u\cdot\mathbf{x}_u)\frac{d^2u}{ds^2} + (\mathbf{x}_u\cdot\mathbf{x}_v)\frac{d^2v}{ds^2} = 0$$

$$\mathbf{k}\cdot\mathbf{x}_v = (\mathbf{x}_{uu}\cdot\mathbf{x}_v)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_{uv}\cdot\mathbf{x}_v)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_{vv}\cdot\mathbf{x}_v)\left(\frac{dv}{ds}\right)^2 + (\mathbf{x}_u\cdot\mathbf{x}_v)\frac{d^2u}{ds^2} + (\mathbf{x}_v\cdot\mathbf{x}_v)\frac{d^2v}{ds^2} = 0$$

Solving for d^2u/ds^2 and d^2v/ds^2 and using the vector identity $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$ gives the equivalent equations

$$(EG - F^2)\frac{d^2u}{ds^2} = (\mathbf{x}_v\times\mathbf{x}_{uu})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_v\times\mathbf{x}_{uv})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_v\times\mathbf{x}_{vv})\cdot(\mathbf{x}_u\times\mathbf{x}_v)\left(\frac{dv}{ds}\right)^2$$

$$(EG - F^2)\frac{d^2v}{ds^2} = (\mathbf{x}_u\times\mathbf{x}_{uu})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\left(\frac{du}{ds}\right)^2 + 2(\mathbf{x}_u\times\mathbf{x}_{uv})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\frac{du}{ds}\frac{dv}{ds} + (\mathbf{x}_u\times\mathbf{x}_{vv})\cdot(\mathbf{x}_v\times\mathbf{x}_u)\left(\frac{dv}{ds}\right)^2$$

Using $\mathbf{N} = \mathbf{x}_u\times\mathbf{x}_v/|\mathbf{x}_u\times\mathbf{x}_v| = \mathbf{x}_u\times\mathbf{x}_v/\sqrt{EG - F^2}$ and the expressions for $[\mathbf{x}_u\mathbf{x}_{uu}\mathbf{N}]$, etc., in the preceding problem gives the required equations.

11.15. If $u(s)$ and $v(s)$ are solutions to the differential equations in the preceding problem such that at some point $s = s_0$, $E_0\left(\frac{du}{ds}\right)_0^2 + 2F_0\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G_0\left(\frac{dv}{ds}\right)_0^2 = 1$, prove that s is a natural parameter along the curve $\mathbf{x} = \mathbf{x}(u(s), v(s))$.

From the preceding problem, $u(s), v(s)$ is a solution to the differential equations if and only if the vector $\frac{d\mathbf{t}}{ds}$, where $\mathbf{t} = \frac{d\mathbf{x}}{ds} = \mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}$, satisfies $\frac{d\mathbf{t}}{ds}\cdot\mathbf{x}_u = 0$ and $\frac{d\mathbf{t}}{ds}\cdot\mathbf{x}_v = 0$, or, equivalently $\frac{d\mathbf{t}}{ds}\cdot\mathbf{U} = 0$ for *every* tangent vector $\mathbf{U}$. But then $\frac{d}{ds}|\mathbf{t}|^2 = \frac{d}{ds}(\mathbf{t}\cdot\mathbf{t}) = 2\frac{d\mathbf{t}}{ds}\cdot\mathbf{t} = 0$ since $\mathbf{t}$ is a tangent vector. Hence integrating gives $|\mathbf{t}|^2 = C = \text{constant}$. But at $s = s_0$,

$$|\mathbf{t}_0|^2 = \left|(\mathbf{x}_u)_0\left(\frac{du}{ds}\right)_0 + (\mathbf{x}_v)_0\left(\frac{dv}{ds}\right)_0\right|^2 = E_0\left(\frac{du}{ds}\right)_0^2 + 2F_0\left(\frac{du}{ds}\right)_0\left(\frac{dv}{ds}\right)_0 + G_0\left(\frac{dv}{ds}\right)_0^2 = 1$$

Thus $|\mathbf{t}|^2 = |d\mathbf{x}/ds|^2 = 1$ for all s, which completes the proof.

11.16. If $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ is a natural representation of a geodesic on a patch $\mathbf{x} = \mathbf{x}(u, v)$ such that $E = E(u)$, $F = 0$ and $G = G(u)$, prove that $\sqrt{G}\cos\theta = C = \text{constant}$, where θ is the angle between the geodesic and the v-parameter curves $u = \text{constant}$, i.e. $\theta = \measuredangle(\mathbf{t}, \mathbf{x}_v)$.

It is easily computed from equation *(10.4)*, page 202, that $\Gamma^2_{11} = 0$, $\Gamma^2_{12} = G_u/2G$, $\Gamma^2_{22} = 0$. Hence the second of the equations *(11.7)* is $\frac{d^2v}{ds^2} + \frac{G_u}{G}\frac{du}{ds}\frac{dv}{ds} = 0$. Since $\frac{d}{ds}\left(G\frac{dv}{ds}\right) = G\frac{d^2v}{ds^2} + G_u\frac{du}{ds}\frac{dv}{ds}$, this is equivalent to $\frac{d}{ds}\left(G\frac{dv}{ds}\right) = 0$. Hence $G\frac{dv}{ds} = C = \text{constant}$. Using $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, we find that

$$G\frac{dv}{ds} \;=\; (\mathbf{x}_v \cdot \mathbf{x}_v)\frac{dv}{ds} \;=\; \left(\mathbf{x}_u \frac{du}{ds} + \mathbf{x}_v \frac{dv}{ds}\right)\cdot \mathbf{x}_v \;=\; \mathbf{t}\cdot\mathbf{x}_v \;=\; |\mathbf{t}|\,|\mathbf{x}_v| \cos \measuredangle(\mathbf{t}, \mathbf{x}_v) \;=\; \sqrt{G}\cos\theta$$

Thus $\sqrt{G}\cos\theta = C = \text{constant}$.

11.17. Prove Theorem 11.10: If $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2 such that $E = E(u)$, $F = 0$ and $G = G(u)$, then

(i) the u-parameter curves $v = \text{constant}$ are geodesics,

(ii) the v-parameter curve $u = u_0$ is a geodesic iff $G_u(u_0) = 0$,

(iii) a curve $\mathbf{x} = \mathbf{x}(u, v(u))$ is a geodesic iff $v = \pm \int \frac{C\sqrt{E}}{\sqrt{G}\sqrt{G - C^2}}\,du$, $C = \text{constant}$.

(i) From equation (11.6), page 233, $(\kappa_g)_{v=\text{constant}} = -\frac{E_v}{2E\sqrt{G}} = 0$. Hence the u-parameter curves $v = \text{constant}$ are geodesics.

(ii) Again from equation (11.6), $(\kappa_g)_{u=u_0} = \frac{G_u(u_0)}{2G(u_0)\sqrt{E(u_0)}}$. Hence $u = u_0$ is a geodesic if and only if $G_u(u_0) = 0$.

(iii) As in the preceding problem, $G\frac{dv}{ds} = C = \text{constant}$. Also, since s equals arc length, $1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right|^2 = E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2$. Substituting $\frac{dv}{ds} = \frac{C}{G}$ gives $1 = E\left(\frac{du}{ds}\right)^2 + \frac{C^2}{G}$ or $\frac{du}{ds} = \pm\frac{\sqrt{G - C^2}}{\sqrt{E}\sqrt{G}}$. Thus $\frac{dv}{du} = \frac{dv}{ds}\Big/\frac{du}{ds} = \pm\frac{C\sqrt{E}}{\sqrt{G}\sqrt{G - C^2}}$, from which the required result follows.

11.18. A patch $\mathbf{x} = \mathbf{x}(u, v)$ is called a *Liouville* patch if $E = G = U + V$ and $F = 0$, where U is a function only of u and V is a function only of v. If $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ is a natural representation of a geodesic on such a patch, prove that

$$U\sin^2\theta - V\cos^2\theta \;=\; C, \qquad C \;=\; \text{constant}$$

where θ is the angle between the geodesic and the u-parameter curves, $v = \text{constant}$; i.e. $\theta = \measuredangle(\mathbf{t}, \mathbf{x}_u)$.

Here $\frac{U'}{2(U+V)} = \Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12}$ and $\frac{V'}{2(U+V)} = \Gamma^1_{12} = -\Gamma^2_{11} = \Gamma^2_{22}$. Substituting into equation (11.4), page 233, and setting $\kappa_g = 0$,

$$-V'\left(\frac{du}{ds}\right)^3 + U'\left(\frac{du}{ds}\right)^2\left(\frac{dv}{ds}\right) - V'\frac{du}{ds}\left(\frac{dv}{ds}\right)^2 + U'\left(\frac{dv}{ds}\right)^3 + 2(U+V)\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right) \;=\; 0$$

or

$$\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]\left(U'\frac{dv}{ds} - V'\frac{du}{ds}\right) + 2(U+V)\left(\frac{du}{ds}\frac{d^2v}{ds^2} - \frac{d^2u}{ds^2}\frac{dv}{ds}\right) \;=\; 0$$

This is equivalent to

$$\frac{d}{ds}\left[\frac{U(dv/ds)^2 - V(du/ds)^2}{(du/ds)^2 + (dv/ds)^2}\right] \;=\; 0$$

Hence

$$\text{(a)} \qquad \left[U\left(\frac{dv}{ds}\right)^2 - V\left(\frac{du}{ds}\right)^2\right]\Big/\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right] \;=\; C = \text{constant}$$

Now $\cos\theta = \frac{\mathbf{t}\cdot\mathbf{x}_u}{|\mathbf{t}|\,|\mathbf{x}_u|} = \frac{(\mathbf{x}_u\,du/ds + \mathbf{x}_v\,dv/ds)\cdot\mathbf{x}_u}{|\mathbf{x}_u|} = |\mathbf{x}_u|\frac{du}{ds}$. Similarly $\sin\theta = \frac{\mathbf{t}\cdot\mathbf{x}_v}{|\mathbf{x}_v|} = |\mathbf{x}_v|\frac{dv}{ds}$. Since s equals arc length,

$$1 = \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left|\mathbf{x}_u\frac{du}{ds} + \mathbf{x}_v\frac{dv}{ds}\right|^2 = (U+V)\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$$

$$= |\mathbf{x}_v|^2\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right] = |\mathbf{x}_u|^2\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$$

Thus $\cos^2\theta = \left(\frac{du}{ds}\right)^2 \Big/ \left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$ and $\sin^2\theta = \left(\frac{dv}{ds}\right)^2 \Big/ \left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$. Substituting in (a) above gives the required result.

11.19. *Liouville's Formula.* Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class ≥ 2 such that the u- and v-parameter curves are orthogonal and let $\mathbf{x} = \mathbf{x}(s) = \mathbf{x}(u(s), v(s))$ be a naturally represented curve C on the patch of class C^2. Let $\mathbf{g}_1 = \mathbf{x}_u/|\mathbf{x}_u|$ and $\mathbf{g}_2 = \mathbf{x}_v/|\mathbf{x}_v|$ be the unit vectors in the direction of the parameter curves and let $\theta = \theta(s)$ be the function along C defined by $\mathbf{t} = (\cos\theta)\mathbf{g}_1 + (\sin\theta)\mathbf{g}_2$, where $\mathbf{t}$ is the unit tangent along C. Prove that the geodesic curvature of C is given by

$$\kappa_g = d\theta/ds + \kappa_1\cos\theta + \kappa_2\sin\theta$$

where κ_1 is the geodesic curvature of the u-parameter curve and κ_2 is the geodesic curvature of the v-parameter curve.

Differentiating $\mathbf{g}_1$ along C and using equations (11.21), page 244, gives

$$\frac{d\mathbf{g}_1}{ds} = \frac{\partial\mathbf{g}_1}{\partial u}\frac{du}{ds} + \frac{\partial\mathbf{g}_1}{\partial v}\frac{dv}{ds} = \frac{d\mathbf{g}_1}{ds_1}\frac{ds_1}{du}\frac{du}{ds} + \frac{d\mathbf{g}_1}{ds_2}\frac{ds_2}{dv}\frac{dv}{ds}$$

$$= \frac{d\mathbf{g}_1}{ds_1}|\mathbf{x}_u|\frac{du}{ds} + \frac{d\mathbf{g}_1}{ds_2}|\mathbf{x}_v|\frac{dv}{ds} = \frac{d\mathbf{g}_1}{ds_1}\cos\theta + \frac{d\mathbf{g}_1}{ds_2}\sin\theta$$

where s_1 is arc length along the u-parameter curves and s_2 is arc length along the v-parameter curves. Similarly, $\frac{d\mathbf{g}_2}{ds} = \frac{d\mathbf{g}_2}{ds_1}\cos\theta + \frac{d\mathbf{g}_2}{ds_1}\sin\theta$. Differentiating $\mathbf{t}$ along C and using the above gives

$$\mathbf{k} = \dot{\mathbf{t}} = (\cos\theta)\frac{d\mathbf{g}_1}{ds} - (\sin\theta)\mathbf{g}_1\frac{d\theta}{ds} + (\sin\theta)\frac{d\mathbf{g}_2}{ds} + (\cos\theta)\mathbf{g}_2\frac{d\theta}{ds}$$

$$= \frac{d\mathbf{g}_1}{ds_1}\cos^2\theta + \left(\frac{d\mathbf{g}_1}{ds_2} + \frac{d\mathbf{g}_2}{ds_1}\right)\cos\theta\sin\theta + \frac{d\mathbf{g}_2}{ds_2}\sin^2\theta + \mathbf{U}\frac{d\theta}{ds}$$

where $\mathbf{U} = -\mathbf{g}_1\sin\theta + \mathbf{g}_2\cos\theta$. From equation (11.2), page 233, and the fact that $\mathbf{g}_1\cdot\frac{d\mathbf{g}_1}{ds_1} = \mathbf{g}_1\cdot\frac{d\mathbf{g}_1}{ds_2} = \mathbf{g}_2\cdot\frac{d\mathbf{g}_2}{ds_1} = \mathbf{g}_2\cdot\frac{d\mathbf{g}_2}{ds_2} = 0$, we have

$$\kappa_g = \mathbf{k}\cdot\mathbf{U} = \mathbf{k}\cdot(-\mathbf{g}_1\sin\theta + \mathbf{g}_2\cos\theta)$$

$$= \left(\mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_1}\right)\cos^3\theta + \left(\mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_2}\right)\cos^2\theta\sin\theta$$

$$- \left(\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_1}\right)\sin^2\theta\cos\theta - \left(\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_2}\right)\sin^3\theta + \frac{d\theta}{ds}$$

Finally we observe that the geodesic curvature along the u-parameter curves is given by $\kappa_1 = \mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_1}$ and the curvature along the v-parameter curve is $\kappa_2 = -\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_2}$ and since $\mathbf{g}_1\cdot\mathbf{g}_2 = 0$ also $\kappa_1 = -\mathbf{g}_1\cdot\frac{d\mathbf{g}_2}{ds_1}$ and $\kappa_2 = \mathbf{g}_2\cdot\frac{d\mathbf{g}_1}{ds_2}$. Thus

$$\kappa_g = d\theta/ds + \kappa_1\cos^3\theta + \kappa_2\cos^2\theta\sin\theta + \kappa_1\sin^2\theta\cos\theta + \kappa_2\sin^3\theta$$

which gives the required result.

GEODESIC COORDINATES

11.20. Let $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, be a curve C of class C^2 on a surface of class ≥ 3. Let $\mathbf{x} = \mathbf{x}(s,t)$ be the family of naturally represented geodesics issuing from C and orthogonal to C, i.e. $\mathbf{x}(0,t) = \mathbf{x}(t)$ and $\mathbf{x}_s(0,t)\cdot\dot{\mathbf{x}}(t) = 0$. Prove that there exists an $\epsilon > 0$ such that $\mathbf{x} = \mathbf{x}(s,t)$ is a regular parametric representation of class C^2 for $-\epsilon < s < \epsilon$ and $a < t < b$.

Let P be an arbitrary point on C and $\mathbf{x} = \mathbf{x}(u, v)$ a patch containing P. Let $u = u_0(t)$, $v = v_0(t)$ be the curve in the parameter plane which maps onto C in a neighborhood of P and let $u = u(s, t)$, $v = v(s, t)$ be the family of curves which map onto the geodesics $\mathbf{x} = \mathbf{x}(s, t)$. From Theorem 11.8, page 234, for all t, $u(s, t)$ and $v(s, t)$ are the unique solutions to the differential equations (*11.7*), page 234, satisfying the initial conditions

$$u(0, t) = u_0(t), \quad v(0, t) = v_0(t), \quad u_s(0, t) = \xi(t), \quad v_s(0, t) = \eta(t)$$

where the $\xi(t)$, $\eta(t)$ are of class C^1 and are uniquely determined by the equations

$$\text{(i)} \quad E\xi^2 + 2F\xi\eta + G\eta^2 = 1$$

$$\text{(ii)} \quad E\xi\frac{du_0}{dt} + F\left(\xi\frac{dv_0}{dt} + \eta\frac{du_0}{dt}\right) + G\eta\frac{dv_0}{dt} = 0 \qquad \text{(iii)} \quad \det\begin{pmatrix} \xi & du_0/dt \\ \eta & dv_0/dt \end{pmatrix} > 0$$

Equation (i) states that initially $|\mathbf{x}_s(0, t)| = 1$; equation (ii) states that $\cos \measuredangle(\mathbf{x}_s(0, v), d\mathbf{x}/dt) = 0$, i.e. the geodesics cut C orthogonally; and equation (iii) determines the orientation of the geodesics $\mathbf{x} = \mathbf{x}(s, t)$. Note that the determinant is different from zero since the geodesics cut C orthogonally. But then from the theory on the dependence on initial conditions of solutions to differential equations, it follows that the functions $u(s, t)$ and $v(s, t)$ have continuous second order derivatives in some neighborhood of C. Also the Jacobian $\partial(u, v)/\partial(s, t)$ is different from zero in a neighborhood of C since it is continuous and at $(0, t)$

$$\left.\frac{\partial(u, v)}{\partial(s, t)}\right|_{(0,t)} = \det\begin{pmatrix} u_s(0, t) & u_t(0, t) \\ v_s(0, t) & v_t(0, t) \end{pmatrix} = \det\begin{pmatrix} \xi & du_0/dt \\ \eta & dv_0/dt \end{pmatrix} \neq 0$$

Thus in a neighborhood of the point P the function $\mathbf{x} = \mathbf{x}(s, t) = \mathbf{x}(u(s, t), v(s, t))$ is a regular parametric representation of class C^2. Since C is compact, there exists an $\epsilon > 0$ such that $\mathbf{x} = \mathbf{x}(s, t)$ is a regular parametric representation of class C^2 for $-\epsilon < s < \epsilon$, $a < t < b$.

11.21. Prove that there exists an $\epsilon > 0$ such that a geodesic polar coordinate system $\mathbf{x} = \mathbf{x}(r, \theta)$ at a point P on a surface of class ≥ 3 is a regular parametric representation of class C^2 for $0 < r < \epsilon$, $-\infty < \theta < \infty$, mapping $0 < r < \epsilon$, $0 \leq \theta < 2\pi$, one-to-one onto a deleted neighborhood of P.

Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch containing P such that $(0, 0)$ maps into P and such that at P, $\mathbf{x}_u = \mathbf{g}_1$ and $\mathbf{x}_v = \mathbf{g}_2$, where $\mathbf{g}_1$ and $\mathbf{g}_2$ are the orthogonormal tangent vectors with respect to which θ is measured. Note that at P, $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$. Now with respect to the patch $\mathbf{x} = \mathbf{x}(u, v)$ consider the differential equations (*11.7*)

$$(a) \qquad \begin{aligned} u'' + \Gamma^1_{11}(u')^2 + \Gamma^1_{12}u'v' + \Gamma^1_{22}(u')^2 &= 0 \\ v'' + \Gamma^2_{11}(u')^2 + \Gamma^2_{12}u'v' + \Gamma^1_{22}(v')^2 &= 0 \end{aligned}$$

with initial conditions

$$(b) \qquad u(0) = 0, \quad v(0) = 0, \quad u'(0) = \xi, \quad v'(0) = \eta$$

From the theory of differential equations, for all ξ, η there exists a unique solution $u(t; \xi, \eta)$, $v = (t; \xi, \eta)$ in a neighborhood of $t = 0$ which has continuous second order derivatives with respect to t, ξ, η. Since the equations are linear homogeneous in the second order derivatives and products of two first derivatives, it follows that for any solution $u(t; \xi, \eta)$, $v(t; \xi, \eta)$, the functions $u(s; \rho\xi, \rho\eta)$, $v(s, \rho\xi, \rho\eta)$ where $t = \rho s$, is also a solution to the differential equation for small ρs and satisfies the initial conditions $u|_{s=0} = u|_{t=0} = 0$, $v|_{s=0} = v|_{t=0} = 0$, $u_s|_{s=0} = u_t|_{t=0} = \rho\xi$ and $v_s|_{s=0} = v_t|_{t=0} = \rho\eta$. Hence $u(t; \xi, \eta) = u(s; \rho\xi, \rho\eta)$ and $v(t; \xi, \eta) = v(s; \rho\xi, \rho\eta)$. In particular, for $s = 1$ we have $u(t; \xi, \eta) = u(1; \rho\xi, \rho\eta)$ and $v(t; \xi, \eta) = v(1; \rho\xi, \rho\eta)$. We now set $x = \rho\xi$, $y = \rho\eta$ and consider the parameter transformation $u = u^*(x, y)$, $v = v^*(x, y)$, where $u^*(x, y) = u(1; x, y)$ and $v^*(x, y) = v(1; x, y)$. The above maps a neighborhood of the origin in the xy plane into a neighborhood of the origin in the uv plane. From the differential equations and initial conditions we see that $u^*(0, 0) = 0$ and $v^*(0, 0) = 0$. Also, at $t = 0$, $x = 0$, $y = 0$ and all ξ, η we have

$$\xi = u_t = u^*_x x_t + u^*_y y_t = u^*_x \xi + u^*_y \eta, \qquad \eta = v_t = v^*_x x_t + v^*_y y_t = v^*_x \xi + v^* \eta$$

Hence $u^*_x = 1$, $u^*_y = 0$, $v^*_x = 0$ and $v^*_y = 1$ and so the Jacobian

$$\left.\frac{\partial(u^*, v^*)}{\partial(x, y)}\right|_{(0,0)} = \det\begin{pmatrix} u^*_x & u^*_y \\ v^*_x & v^*_y \end{pmatrix}_{(0,0)} = 1$$

Since the Jacobian is continuous, it is different from zero in a neighborhood of $(0,0)$. Thus $u = u^*(x,y)$, $v = v^*(x,y)$ is an allowable coordinate transformation of class C^2 mapping a neighborhood of the origin of the xy plane 1-1 onto a neighborhood of the origin of the uv plane. We now consider the mapping $\mathbf{x} = \mathbf{x}^*(x,y) = \mathbf{x}(u^*(x,y), v^*(x,y))$. It is a coordinate patch of class C^2 on the surface in a neighborhood of P mapping $(0,0)$ into P, called a set of *Riemann normal coordinates* at P. It is easily verified that at P, $\mathbf{x}^*_x = \mathbf{x}_u, \mathbf{x}^*_y = \mathbf{x}_v$; and so at P, $E^* = 1, F^* = 0$, $G^* = 1$.

Finally we set $\xi = \cos\phi$ and $\xi = \sin\phi$, so that $x = \rho\cos\phi$ and $y = \rho\sin\phi$, and consider the function $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi) = \mathbf{x}^*(\rho\cos\phi, \rho\sin\phi)$. Clearly $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi)$ is a regular parametric representation of class C^2 for $0 < \rho < \epsilon$ and $-\infty < \phi < \infty$ and maps $0 < \rho < \epsilon$, $0 \leq \phi < 2\pi$ one-to-one onto a deleted neighborhood of P since for these ρ, ϕ, $x = \rho\cos\phi$, $y = \rho\sin\phi$ is one-to-one in a deleted neighborhood of the origin of the xy plane. It remains to show that in fact $\mathbf{x} = \mathbf{x}^{**}(\rho,\phi)$ is the given geodesic polar coordinate system, i.e. $\mathbf{x}^{**}(r,\theta) \equiv \mathbf{x}(r,\theta)$. For a fixed ϕ_0 we have

$$\mathbf{x} \;=\; \mathbf{x}^{**}(\rho,\phi_0) \;=\; \mathbf{x}(u^*(\rho\cos\phi_0, \rho\sin\phi_0),\, v^*(\rho\cos\phi_0, \rho\sin\phi_0))$$

where $$u^*(\rho\cos\phi_0, \rho\sin\phi_0) \;=\; u(1; \rho\cos\phi_0, \rho\sin\phi_0) \;=\; u(\rho; \cos\phi_0, \sin\phi_0)$$

and similarly $v^*(\rho\cos\phi_0, \rho\sin\phi_0) = v(\rho, \cos\phi_0, \sin\phi_0)$. But u and v are solutions of the differential equations (*a*) satisfying the initial condition (*b*) where initially,

$$E(u')^2 + 2Fu'v' + G(v')^2 \;=\; \xi^2 + \eta^2 \;=\; \sin\phi_0^2 + \cos\phi_0^2 \;=\; 1$$

It follows that $\mathbf{x} = \mathbf{x}^*(\rho,\phi_0)$ is the naturally represented geodesic through P in the direction of the tangent vector $\mathbf{x}_u\cos\phi_0 + \mathbf{x}_v\sin\phi_0 = \mathbf{g}_1\cos\phi_0 + \mathbf{g}_2\cos\phi_0$. Since these geodesics are unique it follows that $\mathbf{x}^{**}(r,\theta) \equiv \mathbf{x}(r,\theta)$, which completes the proof.

11.22. Prove that all partial derivatives of the first fundamental coefficients of a set of Riemann normal coordinates at P vanish at P.

Let $\mathbf{x} = \mathbf{x}(x,y)$ be a set of Riemann normal coordinates at P. Then for each θ_0 and $x = r\cos\theta_0$, $y = r\sin\theta_0$, the curve $\mathbf{x} = \mathbf{x}(r) = \mathbf{x}(x(r,\theta_0), y(r,\theta_0))$ is a naturally represented geodesic through P. Hence $x(r,\theta_0)$, $y(r,\theta_0)$ satisfy

$$\ddot{x} + \Gamma^1_{11}(\dot{x})^2 + 2\Gamma^1_{12}\dot{x}\dot{y} + \Gamma^1_{22}(\dot{y})^2 \;=\; 0$$

$$\ddot{y} + \Gamma^2_{11}(\dot{x})^2 + 2\Gamma^2_{12}\dot{x}\dot{y} + \Gamma^2_{22}(\dot{y})^2 \;=\; 0$$

Since $\dot{x} = \dfrac{d}{dr}(r\cos\theta_0) = \cos\theta_0$, $\ddot{x} = 0$, $\dot{y} = \sin\theta_0$ and $\ddot{y} = 0$, it follows that

$$\Gamma^1_{11}\cos^2\theta_0 + 2\Gamma^1_{12}\cos\theta_0\sin\theta_0 + \Gamma^1_{22}\sin^2\theta_0 \;=\; 0$$

$$\Gamma^2_{11}\cos^2\theta_0 + 2\Gamma^2_{12}\cos\theta_0\sin\theta_0 + \Gamma^2_{22}\sin^2\theta_0 \;=\; 0$$

But at P the above is true for all θ_0. Hence at P, $\Gamma^k_{ij} = 0$ for all $i,j,k = 1,2$. We recall further that $E = G = 1$, $F = 0$ at P. Hence from equations (*10.4*), page 202, at P,

$$\Gamma^1_{11} = \tfrac{1}{2}E_x = 0 \qquad \Gamma^1_{12} = \tfrac{1}{2}E_y = 0 \qquad \Gamma^1_{22} = \tfrac{1}{2}(2F_y - G_x) = 0$$

$$\Gamma^2_{11} = \tfrac{1}{2}(2F_x - E_y) = 0 \qquad \Gamma^2_{12} = \tfrac{1}{2}G_x = 0 \qquad \Gamma^2_{22} = \tfrac{1}{2}G_y = 0$$

from which the required result follows.

11.23. Prove Theorem 11.13: If $\mathbf{x} = \mathbf{x}(r,\theta)$ is a set of geodesic polar coordinates at a point P on a surface of sufficiently high class, then

$$\sqrt{G} \;=\; r - \tfrac{1}{6}K(P)r^3 + R(r,\theta)$$

where $\lim_{r\to 0}(R(r,\theta)/r^3) = 0$ and $K(P)$ is the Gaussian curvature at P.

Let $\mathbf{x} = \mathbf{x}^*(x,y)$ be Riemann normal coordinates at P. Then for $r > 0$, and any wedge $\theta_1 < \theta < \theta_2$, where, say, $\theta_2 - \theta_1 \leq \frac{3}{2}\pi$, $x = r\cos\theta$, $y = r\sin\theta$ is an allowable coordinate transformation and from equations (*9.3*), page 172,

$$G \;=\; E^*(x_\theta)^2 + 2F^*x_\theta y_\theta + G^*y_\theta^2 \;=\; r^2(E^*\sin^2\theta - 2F^*\sin\theta\cos\theta + G^*\cos^2\theta)$$

From the preceding problem $E^* = G^* = 1$, $F^* = 0$, $E^*_x = E^*_y = F^*_x = F^*_y = G^*_x = G^*_y = 0$ at P. Using this, it is easily calculated from the above that

(a) $$\lim_{r\to 0} \sqrt{G} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial\sqrt{G}}{\partial r} = 1$$

Also from equation (11.11), page 232, we have $\dfrac{\partial^2\sqrt{G}}{\partial r^2} = -K\sqrt{G}$. Differentiating gives $\dfrac{\partial^3\sqrt{G}}{\partial r^3} = -K\dfrac{\partial\sqrt{G}}{\partial r} - \dfrac{\partial K}{\partial r}\sqrt{G}$. Thus, using (a) above,

(b) $$\lim_{r\to 0} \frac{\partial^2\sqrt{G}}{\partial r^2} = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{\partial^3\sqrt{G}}{\partial r^3} = -K(P)$$

For each θ we can write

$$\sqrt{G} = (\sqrt{G})_0 + \left(\frac{\partial\sqrt{G}}{\partial r}\right)_0 r + \frac{1}{2}\left(\frac{\partial^2\sqrt{G}}{\partial r^2}\right)_0 r^2 + \frac{1}{6}\left(\frac{\partial^3\sqrt{G}}{\partial r^3}\right)_0 r^3 + R(r,\theta)$$

Hence from equations (a) and (b) above,

$$\sqrt{G} = r - K(P)r^3 + R(r,\theta)$$

where $\lim_{r\to 0} (R(r,\theta)/r^3) = 0$, which completes the proof.

SURFACES OF CONSTANT GAUSSIAN CURVATURE

11.24. Suppose R is a region on a patch $\mathbf{x} = \mathbf{x}(u,v)$ on a surface of sufficiently high class. The endpoints of the unit normal $\mathbf{N}$ in R form a set R' on the unit sphere called the spherical image of R. Show that the ratio of the area of R' to the area of R tends to K at a point P where R shrinks down to P.

From equation (9.8), page 174, the element of surface area on the patch is

$$dR = \sqrt{EG - F^2}\, du\, dv = |\mathbf{x}_u \times \mathbf{x}_v|\, du\, dv$$

and the element of surface area on the spherical image is $dR' = |\mathbf{N}_u \times \mathbf{N}_v|\, du\, dv$. From Problem 9.18, page 194, $\mathbf{N}_u \times \mathbf{N}_v = K|\mathbf{x}_u \times \mathbf{x}_v|$. Thus $dR'/dR = K$, which proves the proposition.

11.25. A ruled surface (see Problem 8.4, page 183) is called a *developable* surface if the tangent plane is constant along each ruling. Prove that the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}(s)| = 1$, where $\mathbf{y} = \mathbf{y}(s)$ is a naturally represented base curve, is a developable if and only if $[\dot{\mathbf{y}}\mathbf{g}\dot{\mathbf{g}}] = 0$.

The tangent plane at a point on the ruling $\mathbf{x} = \mathbf{x}(s_0, v)$ is spanned by the vectors $\mathbf{x}_s(s_0, v) = \dot{\mathbf{y}}(s_0) + v\dot{\mathbf{g}}(s_0)$ and $\mathbf{x}_v(s_0, v) = \mathbf{g}(s_0)$. At $v = 0$ these are the vectors $\mathbf{x}_s(s_0, 0) = \dot{\mathbf{y}}(s_0)$ and $\mathbf{x}_v(s_0, 0) = \mathbf{g}(s_0) = \mathbf{x}_v(s_0, v)$. It follows that the tangent plane is the same along a ruling if and only if the three vectors $\dot{\mathbf{y}} + v\dot{\mathbf{g}}$, $\mathbf{g}$ and $\dot{\mathbf{y}}$ are dependent, i.e. if and only if

$$0 = [(\dot{\mathbf{y}} + v\dot{\mathbf{g}})\mathbf{g}\dot{\mathbf{y}}] = \dot{\mathbf{y}} \times (\dot{\mathbf{y}} + v\dot{\mathbf{g}}) \cdot \mathbf{g} = v\dot{\mathbf{y}} \times \dot{\mathbf{g}} \cdot \mathbf{g} = -v[\dot{\mathbf{y}}\mathbf{g}\dot{\mathbf{g}}]$$

which proves the proposition.

11.26. Let $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}(s)| = 1$, $a < s < b$, be a developable surface. Prove that the interval $a < s < b$ can be subdivided into subintervals $s_{i-1} < s < s_i$ such that on each of the subdivisions the surface is either a plane, a cylinder, a cone, or the tangent surface of a curve.

From the preceding problem the vectors $\dot{\mathbf{y}}$, $\mathbf{g}$ and $\dot{\mathbf{g}}$ are dependent. Thus there exist $k_1(s)$, $k_2(s)$ and $k_3(s)$, where $k_1^2 + k_2^2 + k_3^2 \neq 0$, such that $k_1\dot{\mathbf{y}} + k_2\mathbf{g} + k_3\dot{\mathbf{g}} = 0$. Now suppose $k_1 \equiv 0$ in an interval $s_{i-1} < s < s_i$. Then $k_2\mathbf{g} + k_3\dot{\mathbf{g}} = 0$ where $k_2^2 + k_3^2 \neq 0$. Since $|\mathbf{g}| = 1$, $\mathbf{g} \cdot \dot{\mathbf{g}} = 0$. Thus $0 = (k_2\mathbf{g} + k_3\dot{\mathbf{g}}) \cdot \mathbf{g} = k_2|\mathbf{g}|^2$. Hence $k_2 \equiv 0$. But then $k_3 \neq 0$. Hence $\dot{\mathbf{g}} \equiv 0$ or $\mathbf{g} = \text{constant}$. It follows that in this case the surface is a portion of a plane or a cylinder. Now suppose $k_1 \neq 0$ in $s_{i-1} < s < s_i$. Then we can write $\dot{\mathbf{y}} = c_1\mathbf{g} + c_2\dot{\mathbf{g}}$ where $c_1 = -k_2/k_1$ and $c_2 = -k_3/k_1$. Now let $\mathbf{y}^* = \mathbf{y} - c_2\mathbf{g}$. Then $\dot{\mathbf{y}}^* = \dot{\mathbf{y}} - c_2\dot{\mathbf{g}} - \dot{c}_2\mathbf{g} = c_3\mathbf{g}$ where $c_3 = c_1 - \dot{c}_2$. If $c_3 \equiv 0$ in $s_{i-1} < s < s_i$, then $\mathbf{y}^* = \text{constant} = \mathbf{y}_0^*$. Hence the surface is of the form $\mathbf{x} = \mathbf{y}_0^* + (v + c_2)\mathbf{g}$. But this is either the equation of a cone or part of a plane. Finally we have the case where $\dot{\mathbf{y}}^* = c_3\mathbf{g}$ and $c_3 \neq 0$ in some subinterval of $s_{i-1} < s < s_i$. Then $\mathbf{g} = \dot{\mathbf{y}}^*/c_3$ and so the surface is of the form $\mathbf{x} = \mathbf{y} + v\mathbf{g} = \mathbf{y}^* + u\dot{\mathbf{y}}^*$, where $u = (v + c_2)/c_3$, which is the tangent surface of the curve $\mathbf{x} = \mathbf{y}^*(s)$.

11.27. Prove that a surface of sufficiently high class without planar points has constant zero Gaussian curvature if and only if a neighborhood of every point on the surface is a developable surface. *Note.* It then follows from Theorem 11.17, page 241, that a neighborhood of every point on a surface of sufficiently high class without planar points can be mapped isometrically onto a plane if and only if a neighborhood of every point on the surface is a developable surface.

It is easily verified directly that a plane, cylinder, cone, or tangent surface of a curve has constant zero Gaussian curvature. Thus it follows from the preceding problem that if a neighborhood of every point on a surface is a developable surface, then $K = 0$ on the surface.

Another interesting proof of this, is to consider the spherical image of a developable surface. Since the tangent plane is constant along the family of rulings of the surface, its spherical image is either a point (in the case of a plane) or a line. Now let P be a point on the surface, R a region containing the point and R' the spherical image of R. From Problem 11.24 the Gaussian curvature at P in absolute value is equal to the ratio of the area of R' to the area of R as R shrinks down to P. But the spherical image of a developable surface is at best a line. Thus for all R the area of R' is zero. Hence $K = 0$.

Now suppose that $K \equiv 0$ on the surface. Let P be a point on the surface and $\mathbf{x} = \mathbf{x}(u,v)$ a patch containing P. Since $K = \dfrac{LN - M^2}{EG - F^2} \equiv 0$, we have $LN - M^2 \equiv 0$. Since there are no planar points on the patch, it follows that each point is a parabolic point with a single asymptotic direction $du:dv$ satisfying $\text{II} = L\,du^2 + 2M\,du\,dv + N\,dv^2 = (\sqrt{L}\,du + \sqrt{N}\,dv)^2 = 0$ where we used the fact that $\text{II} = 0$ has a single real root and so can factor into a square. The above equation provides a one-parameter family of asymptotic lines in the neighborhood of P which we suppose to have been chosen as the u-parameter curves $v = \text{constant}$. Note that at a parabolic point the asymptotic direction coincides with a principal direction and so the u-parameter curves are also lines of curvature. Now along these curves we have $dv = 0$. Thus from the above differential equation, $\sqrt{L}\,du = 0$. Since $du \neq 0$, $L = 0$. But $LN - M^2 = 0$. Hence also $M = 0$. Since the u-parameter curves cover a neighborhood of P, $L \equiv M \equiv 0$ in the neighborhood. From the Weingarten equations *(10.5)*, page 202, it follows that $\mathbf{N}_u = \mathbf{0}$ and thus $\mathbf{N}$ is constant along each u-parameter curve. It remains to show that the u-parameter curves are straight lines. Since they are also lines of curvature, we have, from the Rodrigues formula, $\mathbf{N}_v = -\kappa\mathbf{x}_u$. Since $\mathbf{N}_u = \mathbf{0}$, $\kappa = 0$ and so the neighborhood of P is a developable surface with the u-parameter curves as rulings along which the tangent plane is constant.

11.28. Prove that at (u,θ) the surface

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + (\log u)\mathbf{e}_3, \qquad u > 0$$

has the same Gaussian curvature as the surface

$$\mathbf{x}^* = (u^*\cos\theta^*)\mathbf{e}_1 + (u^*\sin\theta^*)\mathbf{e}_2 + u^*\mathbf{e}_3, \qquad u^* > 0$$

at $u^* = u$ and $\theta^* = \theta$ but the surfaces are not isometric.

We leave as an exercise for the reader to verify that on $\mathbf{x}$ we have $E = (1 + 1/u^2)$, $F = 0$, $G = u^2$ and $K = -1/(1+u^2)^2$; and on $\mathbf{x}^*$ we have $E^* = 1$, $F^* = 0$, $G^* = 1 + (u^*)^2$ and $K^* = -1/(1+(u^*)^2)^2$. Thus the surfaces have the same Gaussian curvature at corresponding points. Now suppose the surfaces are isometric. Then there exists a parameter transformation $\theta^* = \theta^*(\theta,u)$, $u^* = u^*(\theta,u)$ such that at corresponding points $E = E^*$, $F = F^*$, $G = G^*$ and $K = K^*$. From $K = K^*$ we obtain $1 + (u^*)^2 = 1 + (u)^2$. Then $u^* = \pm u$ or $u^* = \sqrt{-2-u^2}$. Using the transformation properties of the first fundamental coefficients (equations *(9.2)* and *(9.3)*, page 172) and assuming $u^* = \pm u$, it is easily computed that the parameter transformation must also satisfy

$$(a)\ 1 + (1+u^2)(\theta_u^*)^2 = 1 + 1/u^2, \qquad (b)\ \theta_u^*\theta_\theta^* = 0 \quad \text{and} \quad (c)\ (1+u^2)\theta_\theta^2 = u^2$$

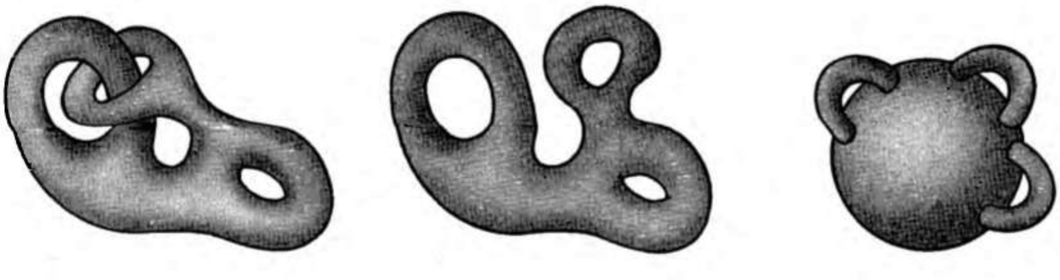
Since u^* is independent of θ, $u_\theta^* \equiv 0$. Since we must have $\partial(\theta^*, u^*)/\partial(\theta, u) \neq 0$, we must have $\theta_\theta^* \neq 0$. Hence from equation (b), $\theta_u^* \equiv 0$. But then it is impossible to satisfy (a) and the theorem is proved.

GAUSS-BONNET THEOREM

11.29. Determine the total curvature of an ellipsoid.

The ellipsoid is homeomorphic to the sphere. Hence the total curvature of the ellipsoid is equal to the total curvature of sphere which is 4π.

11.30. Determine the total curvature of the surface shown in Fig. 11-28(*a*).



(*a*) (*b*) (*c*)

Fig. 11-28

As shown in Fig. 11-28(*b*) and (*c*), the surface is homeomorphic to a sphere with 3 handles. It follows from the formula $\chi = 2(1-p)$, where p is the number of handles of the surface, (see Example 11.11, page 246), that the total curvature of the surface is $2\pi\chi = -8\pi$.

11.31. Determine directly all terms of the Gauss-Bonnet formula (*11.22*), page 244, for the image of the polygon with edges C_1': $\theta = t$, $\phi = \pi/4$, $0 \leq t \leq \pi/2$; C_2': $\theta = \pi/2$, $\phi = t$, $\pi/4 \leq t \leq \pi/2$; C_3': $\theta = \pi/2 - t$, $\phi = \pi/2$, $0 \leq t \leq \pi/2$; C_4': $\theta = 0$, $\phi = \pi/2 - t$, $0 \leq t \leq \pi/4$ on the sphere of radius one

$$\mathbf{x} = (\cos\theta\sin\phi)\mathbf{e}_1 + (\sin\theta\sin\phi)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3$$

See Fig. 11-29.

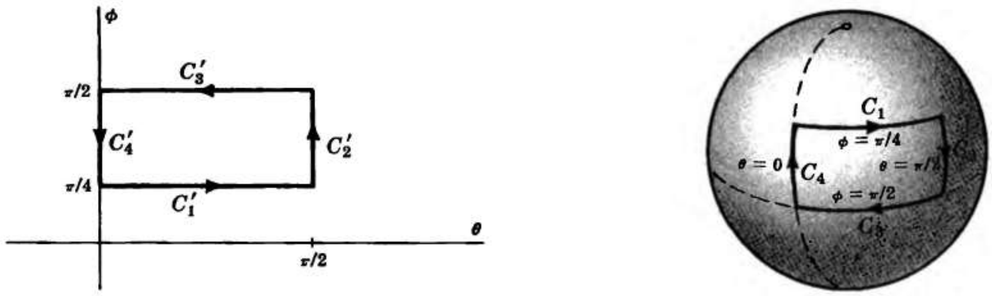


Fig. 11-29

Here $E = \sin^2\phi$, $F = 0$, $G = 1$ and the Gaussian curvature $K = 1$. Hence

(*a*) $$\iint_R K\,dS = \iint_{R'} \sqrt{EG - F^2}\,d\theta\,d\phi = \int_0^{\pi/2}\left[\int_{\pi/4}^{\pi/2} \sin\phi\,d\phi\right] d\theta = \pi\sqrt{2}/4$$

From equation (*11.6*), page 233, $(K_g)_{\phi=\text{constant}} = -\cot\phi$ and so

$$\int_{C_1} \kappa_g\,dS = -\int_0^{\pi/2} \cot(\pi/4)\sqrt{E\left(\frac{d\theta}{dt}\right)^2}\,dt = -\int_0^{\pi/2} \cos(\pi/4)\,dt = -\pi\sqrt{2}/4$$

Since C_2, C_3 and C_4 are geodesics, $\int_{C_2} \kappa_g\,ds = \int_{C_3} \kappa_g\,ds = \int_{C_4} \kappa_g\,ds = 0$. Thus

(*b*) $$\int_C \kappa_g\,ds = \int_{C_1} \kappa_g\,ds = -\pi\sqrt{2}/4$$

Finally, since the parameter curves are orthogonal,

(*c*) $$\sum_{i=1}^{4} \alpha_i = 4(\pi/2) = 2\pi$$

11.32. Prove that a surface has constant zero Gaussian curvature if in the neighborhood of each point there exist two families of geodesics which intersect at a constant angle.

Let P be an arbitrary point on the surface and C a quadrilateral made up of geodesics and containing P in its interior. Applying the Gauss-Bonnet formula gives $\iint_R K\,dS = 2\pi - \sum_{i=1}^{4} \alpha_i$. Since the geodesics intersect at a constant angle, $\sum_i \alpha_i = 2\pi$. Thus $\iint_R K\,dS = 0$. Since R can be chosen arbitrarily small, it follows that $K(P) = 0$, which proves the proposition.

11.33. Prove that on a surface with $K < 0$, a geodesic cannot have a multiple point as shown in Fig. 11-30(a), nor can two geodesics have more than one intersection as shown in Fig. 11-30(b), assuming the geodesics bound a simply connected domain.

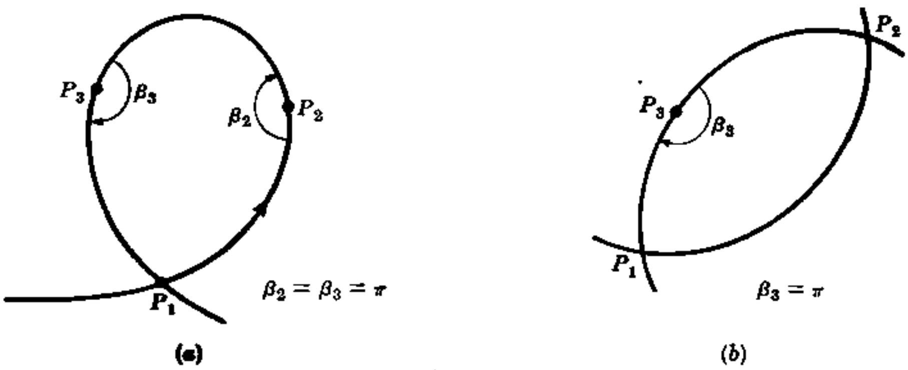


Fig. 11-30

Suppose geodesics having either of the two properties exist. By adding additional vertices to the geodesics, as shown in the figure, both cases become geodesic triangles, for which the Gauss-Bonnet formula is $\iint_R K\,dS = \sum_{i=1}^{3} \beta_i - \pi$ where β_i are the interior angles of the triangles. But in both cases $\sum_{i=1}^{3} \beta_i > \pi$, which is impossible since $K < 0$.

Supplementary Problems

11.34. If S is a compact surface and $\mathbf{f}$ is a regular differentiable mapping of S onto a surface S^*, prove that S^* is compact.

11.35. Prove Theorem 11.2: If $\mathbf{f}$ is a regular differentiable mapping of class C^r of S into S^* and $\mathbf{x} = \mathbf{x}(t)$ is a regular curve C of class C^r on S, then $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(t))$ is a regular curve of class C^r on S^*.

11.36. Show that the stereographic projection of a sphere onto a plane (see Example 11.1, page 227) is a conformal mapping.

11.37. Prove that a 1-1 regular differentiable mapping of a surface S onto a surface S^* is a 1-1 bicontinuous (topological) mapping of S onto S^*.

11.38. Prove that a mapping $\mathbf{f}$ of a surface S into a surface S^* is a local isometry if and only if for every patch $\mathbf{x} = \mathbf{x}(u,v)$ on S we have $E = E^*, F = F^*$ and $G = G^*$ where E, F, G are the first fundamental coefficients on $\mathbf{x} = \mathbf{x}(u,v)$ and E^*, G^*, F^* are the first fundamental coefficients on $\mathbf{x}^* = \mathbf{f}(\mathbf{x}(u,v))$.

11.39. Show that the differential equations of the geodesics on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ are

$$(1+p^2+q^2)\ddot{u} + pr\dot{u}^2 + 2ps\dot{u}\dot{v} + pt\dot{v}^2 = 0$$

$$(1+p^2+q^2)\ddot{v} + qr\dot{u}^2 + 2qs\dot{u}\dot{v} + qt\dot{v}^2 = 0$$

where $p = f_u$, $q = f_v$, $v = f_{uu}$, $s = f_{uv}$, $t = f_{vv}$.

11.40. Find the geodesics on the plane by solving equations (*11.7*), page 234, in polar coordinates.

11.41. Show that the solutions to the equation

$$d\theta = \frac{Ca\,dr}{r\sqrt{r^2-C^2}\sqrt{a^2-(r-b)^2}}$$

where $r = b + a\sin\phi$, are geodesics on the torus

$$\mathbf{x} = (b + a\sin\phi)(\cos\theta)\mathbf{e}_1 + (b + a\sin\phi)(\sin\theta)\mathbf{e}_2 + (a\cos\phi)\mathbf{e}_3$$

11.42. Show that the geodesics on a Liouville surface (see Problem 11.18) are given by

$$\int (U-C)^{-1/2}\,du \pm \int (V+C)^{-1/2}\,dv = \text{constant}, \qquad C = \text{constant}$$

11.43. Let $\mathbf{x} = \mathbf{x}(u,v)$ be a patch on a surface of class ≥ 3 such that the parameter curves are orthogonal. Prove that

$$K = \frac{d(\kappa_g)_1}{ds_1} - \frac{d(\kappa_g)_2}{ds_2} - (\kappa_g)_1^2 - (\kappa_g)_2^2$$

where $(\kappa_g)_1$ and $(\kappa_g)_2$ are the geodesic curvatures along the u- and v-parameter curves respectively and s_1 and s_2 are natural parameters along the respective u- and v-parameter curves.

11.44. Let P and Q be two points on a geodesic $v = \text{constant}$ of a coordinate patch of geodesic coordinates $\mathbf{x} = \mathbf{x}(u,v)$. Prove that of all regular arcs on the patch joining P to Q, the geodesic containing P and Q is the one of shortest length.

11.45. Prove that the surface of revolution

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + f(v)\mathbf{e}_3$$

where $u = C_1\cos(v/a) + C_2\sin(v/a)$ and $f(v) = \int \sqrt{1-(du/dv)^2}\,dv$ is a surface of constant positive Gaussian curvature $K = 1/a^2$ for all C_1, C_2. For what values of C_1 and C_2 is the surface a sphere? *Ans.* $C_1 = a, C_2 = 0$ or $C_1 = 0, C_2 = a$.

11.46. Prove that the surface of revolution

$$\mathbf{x} = u(\cos\theta)\mathbf{e}_1 + u(\sin\theta)\mathbf{e}_2 + f(v)\mathbf{e}_3$$

where $u = C_1e^{v/a} + C_2e^{-v/a}$ and $f(v) = \int \sqrt{1-(du/dv)^2}\,dv$ is a surface constant negative Gaussian curvature $K = -1/a^2$.

11.47. Prove that $\mathbf{x} = 2(\tanh(r/2)\cos\theta)\mathbf{e}_1 + 2(\tan(r/2)\sin\theta)\mathbf{e}_2$

is a set of geodesic polar coordinates at the origin of the hyperbolic plane. See Example 11.9, page 242.

11.48. Determine the intrinsic distance from the origin to a point P in the hyperbolic plane.
Ans. $D(0,P) = \tanh(|P|/2)$.

11.49. Let $\mathbf{x} = \mathbf{y}(s)$ be a naturally represented curve C without points of inflection. Prove that the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\,\mathbf{n}(s)$, where $\mathbf{n}$ is the unit principal normal to C, is a developable surface if and only if $\mathbf{x} = \mathbf{y}(s)$ is a plane curve.

11.50. Prove that a curve $\mathbf{x} = \mathbf{y}(s)$ on orientable surface S is a line of curvature on S if and only if the ruled surface $\mathbf{x} = \mathbf{y}(s) + v\mathbf{N}(s)$, where $\mathbf{N}$ is the normal to S, is a developable surface.

11.51. Let $\mathbf{x} = \mathbf{y}(s) + v\mathbf{g}(s)$, $|\mathbf{g}| = 1$, be a developable surface such that $\mathbf{g} \cdot \mathbf{y} = 0$ and let $\phi(s) = \measuredangle(\mathbf{g}, \mathbf{n})$ where $\mathbf{n}$ is the principal normal to $\mathbf{x} = \mathbf{y}(s)$. Prove that $\phi = -\tau$ where τ is the torsion along $\mathbf{x} = \mathbf{y}(s)$.

11.52. If S is a surface with constant Gaussian curvature $K \neq 0$, prove that the area of a geodesic polygon is determined by its interior angles.

11.53. Let C_n, $n = 1, 2, \ldots$, be an infinite sequence of geodesic triangles which shrink to a point P as $n \to \infty$. Prove that $K(P) = \lim_{n \to \infty} \dfrac{\sum_{i=1}^{3} \beta_{in} - \pi}{A_n}$ where A_n is the area of C_n, and β_{in} are its interior angles.

11.54. Let S be a sphere with p handles. Prove that there exists a point P on S such that (a) $K(P) > 0$ if $p = 0$, (b) $K(P) = 0$ if $p = 1$, (c) $K(P) < 0$ if $p > 1$.

11.55. Let S be a surface with Gaussian curvature $K < 0$. Let P_1, P_2, P_3, P_4 be the vertices of a geodesic quadrilateral with simply connected interior such that the lengths of two opposite sides P_1P_2 and P_3P_4 are equal and perpendicular to the third side P_2P_3. Prove that the interior angles at P_1 and P_4 are acute.

11.56. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a set of geodesic coordinates such that the u-parameters are natural representations of geodesics. Prove that if C is a naturally represented geodesic on the patch and $\theta(s)$ is the function defined by $\mathbf{t} = (\cos\theta)\mathbf{g}_1 + (\sin\theta)\mathbf{g}_2$ where $\mathbf{g}_1$ and $\mathbf{g}_2$ are unit vectors in the direction of the u- and v-parameter curves respectively and $\mathbf{t}$ is the unit tangent to C, then $\dfrac{d\theta}{ds} + \dfrac{\partial\sqrt{G}}{\partial u}\dfrac{dv}{ds} = 0$ along C.

11.57. Using the results of the preceding problem, derive the Gauss-Bonnet formula for a geodesic triangle by taking one of the vertices as the center of a set of geodesic polar coordinates.