If the u- and v-parameter curves on a patch are themselves conjugate families of curves, then (9.29) must be satisfied at each point by $du = 1$, $dv = 0$ and $\delta u = 0$, $\delta v = 1$. Substituting into (9.29) gives $M = 0$. The converse is also true. That is, we have

Theorem 9.20. The u- and v-parameter curves on a patch are conjugate families of curves if and only if at each point $M = 0$.

Note, as a consequence of the above theorem and the corollary to Theorem 9.12 we have

Corollary: The u- and v-parameter curves on a patch without umbilical points are orthogonal and conjugate families of curves if and only if they are lines of curvature.

Example 9.14.

Consider the surface (see Example 9.11)

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$$

Here $L = N = 2(u^2+v^2+1)^{-1/2}$, $M = 0$. Observe that the surface is already represented by conjugate families of parameter curves, since $M = 0$ everywhere. Now consider the family of curves in the parameter plane given by $f(u,v) = u^2 + v^2 = C_1^2$. Since $df = f_u\,du + f_v\,dv = 2u\,du + 2v\,dv = 0$ we have $du : dv = f_v : -f_u = -v : u$. Using (9.29) and dividing by $2(1+u^2+v^2)^{-1/2}$ gives $-v\,\delta u + u\,\delta v = 0$, which has as solutions the family of lines $u = C_2 v$. The two families of curves $u^2 + v^2 = C_1^2$ and $u = C_2 v$ define two conjugate families of curves on the surface which are in fact the lines of curvature.

Solved Problems

FIRST FUNDAMENTAL FORM. ARC LENGTH. SURFACE AREA

9.1. Show that the first fundamental form on the surface of revolution

$$\mathbf{x} = f(t)(\cos\theta)\mathbf{e}_1 + f(t)(\sin\theta)\mathbf{e}_2 + g(t)\,\mathbf{e}_3$$

is

$$\mathrm{I} = f^2\,d\theta^2 + (f'^2 + g'^2)\,dt^2$$

$$\mathbf{x}_\theta = -(f\sin\theta)\mathbf{e}_1 + (f\cos\theta)\mathbf{e}_2, \qquad \mathbf{x}_t = (f'\cos\theta)\mathbf{e}_1 + (f'\sin\theta)\mathbf{e}_2 + g'\mathbf{e}_3$$

$$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = f^2, \quad F = \mathbf{x}_\theta\cdot\mathbf{x}_t = 0, \quad G = \mathbf{x}_t\cdot\mathbf{x}_t = f'^2 + g'^2$$

from which the result follows.

9.2. Find the length of the arc $u = e^{\theta(\cot\beta)/\sqrt{2}}$, $\theta = \theta$, $0 \leq \theta \leq \pi$, $\beta =$ constant, on the cone

$$\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$$

$$E = \mathbf{x}_\theta\cdot\mathbf{x}_\theta = u^2, \quad F = \mathbf{x}_\theta\cdot\mathbf{x}_u = 0, \quad G = \mathbf{x}_u\cdot\mathbf{x}_u = 2, \quad du/d\theta = u(\cot\beta)/\sqrt{2}, \quad d\theta/d\theta = 1.$$

$$\begin{aligned} S &= \int_0^\pi \left[E\left(\frac{d\theta}{d\theta}\right)^2 + 2F\frac{d\theta}{d\theta}\frac{du}{d\theta} + G\left(\frac{du}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \int_0^\pi [u^2 + (\cot^2\beta)u^2]^{1/2}\,d\theta = \sqrt{1+\cot^2\beta}\int_0^\pi u\,d\theta \\ &= \sqrt{1+\cot^2\beta}\int_0^\pi e^{\theta(\cot\beta)/\sqrt{2}}\,d\theta = \frac{\sqrt{2}}{\cos\beta}\left(e^{\pi(\cot\beta)/\sqrt{2}} - 1\right) \end{aligned}$$

which is the required result.

9.3. Show that the curve in Problem 9.2 above intersects the generating lines $\theta = \text{constant}$ on the cone at a constant angle β.

$$\cos \measuredangle\left(\frac{d\mathbf{x}}{d\theta}, \mathbf{x}_u\right) = \frac{(d\mathbf{x}/d\theta)\cdot\mathbf{x}_u}{|d\mathbf{x}/d\theta|\,|\mathbf{x}_u|} = \frac{(\mathbf{x}_\theta(d\theta/d\theta) + \mathbf{x}_u(du/d\theta))\cdot\mathbf{x}_u}{|\mathbf{x}_\theta(d\theta/d\theta) + \mathbf{x}_u(du/d\theta)|\,|\mathbf{x}_u|} = \frac{F + G(du/d\theta)}{\sqrt{E + F(du/d\theta) + G(du/d\theta)^2}\,\sqrt{G}}$$

$$= \frac{\sqrt{2}\,(du/d\theta)}{\sqrt{u^2 + 2(du/d\theta)^2}} = \frac{u \cot\beta}{\sqrt{u^2 + u^2\cot^2\beta}} = \frac{\cot\beta}{\sqrt{1+\cot^2\beta}} = \cos\beta$$

which is the required result.

9.4. If the first fundamental form on a patch is of the form $\mathrm{I} = du^2 + f(u,v)\,dv^2$, prove that the v-parameter curves cut off equal segments from all u-parameter curves. Note, since $F = 0$, the parameter curves are also orthogonal. In this case the v-parameter curves are said to be *parallel*, as shown in Fig. 9-16.

The distance along a u-parameter curve $u = u$, $v = v_0$ between $u = u_1$ and $u = u_2$, $u_1 < u_2$, is

$$d = \int_{u_1}^{u_2} [(du/du)^2 + f(u, v_0)(dv/du)^2]^{1/2}\,du$$

Since $du/du = 1$ and $dv/du = 0$, we have

$$d = \int_{u_1}^{u_2} du = u_2 - u_1$$

which shows that d is the same for each $v = v_0$.

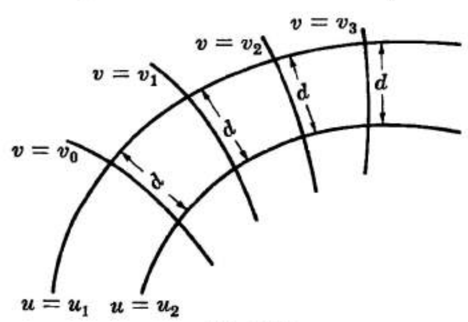

Fig. 9-16

9.5. Show that for a surface represented by $f(x_1, x_2, x_3) = C$, we have $f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 = 0$ where $d\mathbf{x} = (dx_1, dx_2, dx_3)$ is an arbitrary tangent vector to the surface at $P(x_1, x_2, x_3)$.

Suppose $\mathbf{x} = x_1(t)\,\mathbf{e}_1 + x_2(t)\,\mathbf{e}_2 + x_3(t)\,\mathbf{e}_3$ is a regular curve on the surface through $P(x_1, x_2, x_3)$. Then for all t, $f(x_1(t), x_2(t), x_3(t)) = C$. Hence $df/dt = f_{x_1}(dx_1/dt) + f_{x_2}(dx_2/dt) + f_{x_3}(dx_3/dt) = 0$. Namely, if $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is an arbitrary vector in the tangent plane at P, then $f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 = 0$.

9.6. Show that the parameter transformation $u = \sinh t$, $\phi = \theta$, $-\infty < t < \infty$, $0 < \theta < 2\pi$, between the points in the parameter plane of the surface of revolution M_1,

$$\mathbf{x} = (\cosh t \cos\theta)\mathbf{e}_1 + (\cosh t \sin\theta)\mathbf{e}_2 + t\mathbf{e}_3, \quad 0 < \theta < 2\pi$$

and the points in the parameter plane of the right conoid M_2,

$$\mathbf{y} = (u\cos\phi)\mathbf{e}_1 + (u\sin\phi)\mathbf{e}_2 + \phi\mathbf{e}_3, \quad 0 < \phi < 2\pi$$

determines a 1-1 correspondence between the points on the surfaces themselves such that the first fundamental forms agree on corresponding vectors.

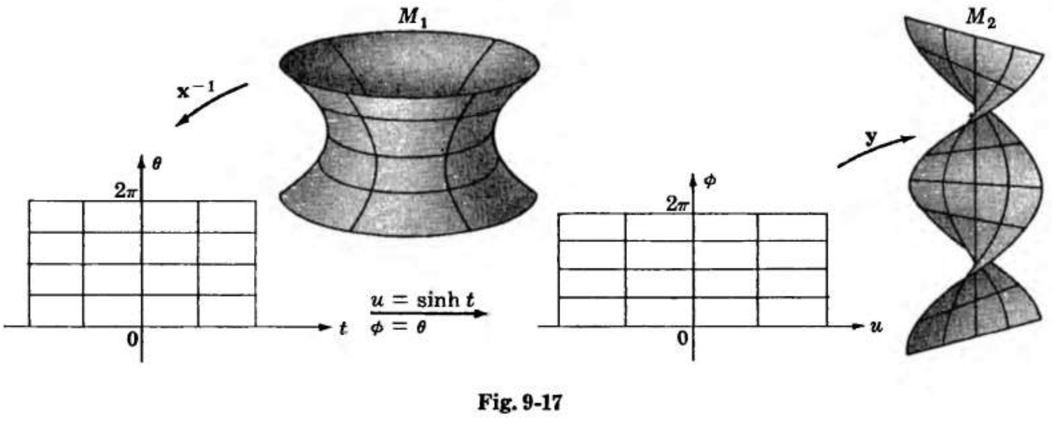

Fig. 9-17

Since $\mathbf{x}$ and $\mathbf{y}$ are both 1-1 mappings of their corresponding domains onto the surfaces M_1 and M_2 and $u = \sinh t$, $\phi = \theta$, defines a 1-1 correspondence between the domains, it follows that the composite mapping $\mathbf{x}^{-1}$ followed by $u = \sinh t$, $\phi = \theta$, followed by $\mathbf{y}$ is a 1-1 mapping of the surface M_1 onto M_2.

Now, at the point $\mathbf{x}(t, \theta)$ on M_1 we have $E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = \cosh^2 t$, $F = \mathbf{x}_\theta \cdot \mathbf{x}_t = 0$, $G = \mathbf{x}_t \cdot \mathbf{x}_t = \cosh^2 t$ and $\mathrm{I} = \cosh^2 t(d\theta^2 + dt^2)$. At $\mathbf{y}(u, \phi)$ on M_2, $E^* = \mathbf{y}_\phi \cdot \mathbf{y}_\phi = u^2 + 1$, $F^* = \mathbf{y}_\phi \cdot \mathbf{y}_u = 0$, $G^* = \mathbf{y}_u \cdot \mathbf{y}_u = 1$ and $\mathrm{I}^* = (u^2+1)\,d\phi^2 + du^2$, but $u = \sinh t$, $\phi = \theta$, $d\phi = d\theta$, $du = \cosh t\,dt$. Hence

$$\mathrm{I}^* \;=\; (\sinh^2 t + 1)\,d\theta^2 + \cosh^2 t\,dt^2 \;=\; \cosh^2 t(d\theta^2 + dt^2) \;=\; \mathrm{I}$$

which is the required result.

SECOND FUNDAMENTAL FORM

9.7. Show that the surface

$$\mathbf{x} \;=\; u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^3)\mathbf{e}_3$$

is elliptic where $v > 0$, hyperbolic where $v < 0$, and parabolic for $v = 0$.

$$\mathbf{x}_u \;=\; \mathbf{e}_1 + 2u\mathbf{e}_3, \qquad \mathbf{x}_v \;=\; \mathbf{e}_2 + 3v^2\mathbf{e}_3$$

$$\mathbf{N} \;=\; \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} \;=\; (4u^2 + 9v^4 + 1)^{-1/2}(-2u\mathbf{e}_1 - 3v^2\mathbf{e}_2 + \mathbf{e}_3)$$

$$\mathbf{x}_{uu} \;=\; 2\mathbf{e}_3, \quad \mathbf{x}_{uv} \;=\; 0, \quad \mathbf{x}_{vv} \;=\; 6v\mathbf{e}_3$$

$$L \;=\; \mathbf{x}_{uu} \cdot \mathbf{N} \;=\; 2(4u^2 + 9v^4 + 1)^{-1/2}, \qquad M \;=\; \mathbf{x}_{uv} \cdot \mathbf{N} \;=\; 0$$

$$N \;=\; \mathbf{x}_{vv} \cdot \mathbf{N} \;=\; 6v(4u^2 + 9v^4 + 1)^{-1/2}, \qquad LN - M^2 \;=\; \frac{12v}{(4u^2 + 9v^4 + 1)}$$

Since $(4u^2 + 9v^2 + 1) > 0$ for all (u, v), we have $LN - M^2 > 0$ for $v > 0$, $LN - M^2 < 0$ where $v < 0$, and $LN - M^2 = 0$ where $v = 0$. Thus the surface is elliptic for $v > 0$, hyperbolic for $v < 0$, and since $L \neq 0$ for all (u, v), parabolic for $v = 0$.

9.8. Show that the surface in Problem 9.7 lies on both sides of the tangent plane in every neighborhood of the parabolic point $P(0, 0)$.

At $u = 0$, $v = 0$ we have $\mathbf{x}_u = \mathbf{e}_1$ and $\mathbf{x}_v = \mathbf{e}_2$; hence the tangent plane at $P(0, 0)$ is the coordinate plane $x_3 = 0$. The v-parameter curve $u = 0$ is the cubic $\mathbf{x} = v\mathbf{e}_2 + v^3\mathbf{e}_3$ which lies on both sides of the tangent plane in every neighborhood of $P(0, 0)$ as shown in Fig. 9-18.

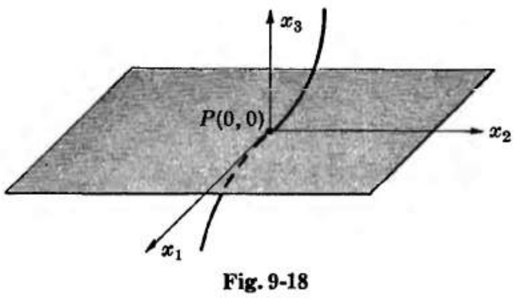

Fig. 9-18

9.9. Prove that all points on the tangent surface of a curve are parabolic.

The tangent surface to the curve $\mathbf{y} = \mathbf{y}(s)$ is $\mathbf{x} = \mathbf{y}(s) + u\mathbf{t}(s)$. Here

$$\mathbf{x}_s \;=\; \dot{\mathbf{y}} + u\dot{\mathbf{t}} \;=\; \mathbf{t} + u\kappa\mathbf{n}, \qquad \mathbf{x}_u \;=\; \mathbf{t}$$

$$\mathbf{N} \;=\; \frac{\mathbf{x}_u \times \mathbf{x}_s}{|\mathbf{x}_u \times \mathbf{x}_s|} \;=\; \frac{u\kappa\mathbf{b}}{|u\kappa|}, \qquad \mathbf{x}_{ss} \;=\; \dot{\mathbf{t}} + u\kappa\dot{\mathbf{n}} + u\dot{\kappa}\mathbf{n} \;=\; -u\kappa^2\mathbf{t} + (u\dot{\kappa} + \kappa)\mathbf{n} + u\kappa\tau\mathbf{b}$$

$$\mathbf{x}_{su} \;=\; \dot{\mathbf{t}} \;=\; \kappa\mathbf{n}, \quad \mathbf{x}_{uu} \;=\; \mathbf{0}, \quad L \;=\; \mathbf{x}_{ss} \cdot \mathbf{N} \;=\; |u\kappa|\tau, \quad M \;=\; \mathbf{x}_{su} \cdot \mathbf{N} \;=\; 0, \quad N \;=\; \mathbf{x}_{uu} \cdot \mathbf{N} \;=\; 0$$

Hence at all points $LN - M^2 = 0$, which is the required result.

9.10. Show that every point on the surface of revolution

$$\mathbf{x} \;=\; f(t)(\cos\theta)\mathbf{e}_1 + f(t)(\sin\theta)\mathbf{e}_2 + t\mathbf{e}_3, \qquad f(t) > 0$$

is a parabolic point if and only if the surface is a right circular cylinder $f = a$, or a cone $f = at + b$, $a = \text{constant} \neq 0$, $b = \text{constant}$.

It can be computed that $L = \frac{-f}{[1+f'^2]^{1/2}}$, $M = 0$, $N = \frac{f''}{[1+f'^2]^{1/2}}$. Hence

$$LN - M^2 = \frac{-ff''}{1+f'^2}$$

Since $f > 0$, $LN - M^2 = 0$ if and only if $f'' = 0$. That is, if and only if $f = a \neq 0$ or $f = at + b$, $a \neq 0$, which completes the proof.

9.11. Prove that a surface lies on one side of the tangent plane in a neighborhood of an elliptic point.

Let $P = \mathbf{x}(u,v)$ be an elliptic point, and $Q = \mathbf{x}(u+du, v+dv)$ a neighboring point. We recall that $d = \mathbf{PQ}\cdot\mathbf{N} = \frac{1}{2}\mathrm{II} + o(du^2+dv^2)$, where $\frac{1}{2}\mathrm{II} = \frac{1}{2}(L\,du^2 + 2M\,du\,dv + N\,dv^2)$ is an elliptic paraboloid in (du, dv) maintaining the same sign for all (du, dv) and equal to zero if and only if $du = 0$, $dv = 0$. Without loss of generality we can assume $\frac{1}{2}\mathrm{II} \geqq 0$. Now let $du = r\cos\theta$, $dv = r\sin\theta$ and consider the quantity

$$\frac{1}{2}\frac{\mathrm{II}}{du^2+dv^2} = \frac{1}{2}\frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{du^2+dv^2} = \frac{1}{2}(L\cos^2\theta + 2M\cos\theta\sin\theta + N\sin^2\theta)$$

But this is just $\frac{1}{2}\mathrm{II}$ evaluated on the unit circle $du = \cos\theta$, $dv = \sin\theta$. Since $\frac{1}{2}\mathrm{II}$ is continuous and greater than zero on the circle, $\frac{1}{2}\mathrm{II}/(du^2+dv^2)$ takes on a minimum $m > 0$. Now select ϵ such that $\frac{o(du^2+dv^2)}{du^2+dv^2} < m$ for $du^2+dv^2 < \epsilon^2$. But then

$$\frac{d}{du^2+dv^2} = \frac{1}{2}\frac{\mathrm{II}}{du^2+dv^2} + \frac{o(du^2+dv^2)}{du^2+dv^2} > 0 \qquad \text{for } 0 < du^2+dv^2 < \epsilon^2$$

It follows that $d \geqq 0$ or Q lies on the same side of the tangent plane for $du^2+dv^2 < \epsilon^2$, which completes the proof.

NORMAL CURVATURE. GAUSSIAN AND MEAN CURVATURE

9.12. Find the normal curvature vector $\mathbf{k}_n$ and normal curvature κ_n of the curve $u = t^2$, $v = t$ on the surface $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$ at the point $t = 1$.

$E = 1+4u^2$, $F = 4uv$, $G = 1+4v^2$, $\mathbf{N} = (4u^2+4v^2+1)^{-1/2}(-2u\mathbf{e}_1 - 2v\mathbf{e}_2 + \mathbf{e}_3)$, $L = 2(4u^2+4v^2+1)^{-1/2}$, $M = 0$, $N = 2(4u^2+4v^2+1)^{-1/2}$, $du/dt = 2t$, $dv/dt = 1$.

At $t = 1$: $u = 1$, $v = 1$, $E = 5$, $F = 4$, $G = 5$, $\mathbf{N} = -1/3[2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3]$, $L = 2/3$, $M = 0$, $N = 2/3$, $du/dt = 2$, $dv/dt = 1$. Hence

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2} = \frac{10}{123}$$

and $\mathbf{k}_n = \kappa_n\mathbf{N} = -\frac{10}{369}[2\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3]$, which are the required results.

9.13. Let L be a tangent line to a surface at a point P in a direction in which $\kappa_n \neq 0$ (i.e. a nonasymptotic direction). Prove that the osculating circles of all curves through P tangent to L lie on a sphere.

Let C be a curve through P tangent to L. Since $\kappa_n = \mathbf{k}\cdot\mathbf{N} \neq 0$, it follows that $\mathbf{k} \neq 0$ and we can write $\kappa_n = \kappa(\mathbf{n}\cdot\mathbf{N}) = \kappa\cos\alpha$ where, by selecting $\mathbf{N}$ so that $\kappa_n > 0$ and $\mathbf{n}$ in the direction of $\mathbf{k}$, we have $\kappa_n > 0$, $\kappa > 0$, and $0 \leqq \alpha = \measuredangle(\mathbf{n},\mathbf{N}) \leqq \pi/2$. Now let $\rho = 1/\kappa$, $R = 1/\kappa_n$; then $\rho = R\cos\alpha$ where R = constant and ρ is the radius of curvature of the osculating circle to C. It follows that the osculating circle is the intersection of the osculating plane and the sphere of radius R tangent to the tangent plane at P as shown in Fig. 9-19, which is the required result.

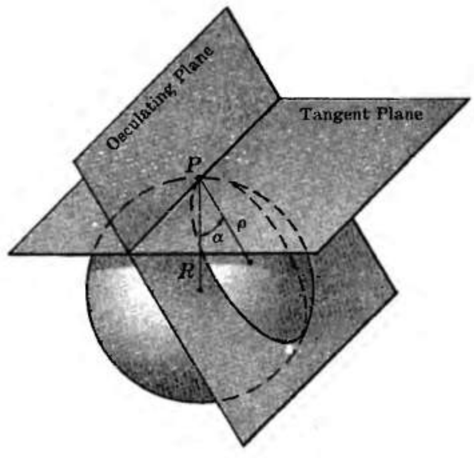

Fig. 9-19

9.14. Prove that the discriminant of the equation

$$(EG-F^2)\kappa^2 - (EN+GL-2FM)\kappa + (LN-M^2) = 0$$

is greater than or equal to zero, and equal to zero if and only if $L/E = M/F = N/G$.

The discriminant is $(EN+GL-2FM)^2 - 4(EG-F^2)(LN-M^2)$, which can be shown to be identically equal to

$$4\left(\frac{EG-F^2}{E^2}\right)(EM-FL)^2 + \left[EN-GL-\frac{2F}{E}(EM-FL)\right]^2$$

Hence the discriminant is greater than or equal to zero. Since $EG-F^2>0$, the above is zero if and only if $EM-FL=0$ and $EN-GL-\frac{2F}{E}(EM-FL)=0$, or if and only if $EM-FL=0$ and $EN-GL=0$ or if and only if $L/E = M/F = N/G$.

9.15. Prove that at every point P on a surface there is a paraboloid such that the normal curvature of the surface at P in any direction is the same as the paraboloid.

Suppose the surface is translated and rotated so that P is at the origin and the tangent plane is the x_1x_2 plane. Then a neighborhood of P can be represented by

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$$

where $\mathbf{x}(0,0)=\mathbf{0}$, $\mathbf{x}_u(0,0)=\mathbf{e}_1$, $\mathbf{x}_v(0,0)=\mathbf{e}_2$. From Taylor's theorem,

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + \tfrac{1}{2}(au^2+2buv+cv^2)\mathbf{e}_3 + \mathbf{o}(u^2+v^2)$$

where $\mathbf{x}_{uu}(0,0) = a\mathbf{e}_3$, $\mathbf{x}_{uv}(0,0) = b\mathbf{e}_3$ and $\mathbf{x}_{vv}(0,0) = c\mathbf{e}_3$, $\mathbf{N}(0,0) = \mathbf{e}_3$, and hence $\kappa_n = \dfrac{a\,du^2 + 2b\,du\,dv + c\,dv^2}{du^2+dv^2}$. The surface represented by the approximation

$$\mathbf{x}^* = u\mathbf{e}_1 + v\mathbf{e}_2 + \tfrac{1}{2}(au^2+2buv+cv^2)\mathbf{e}_3$$

is a paraboloid tangent to the x_1x_2 plane at $u=0$, $v=0$ and such that $\kappa_n^* = \kappa_n$, which is the required result.

9.16. Prove Theorem 9.5. That is, prove that κ_0 is a principal curvature with principal direction $du_0 : dv_0$ if and only if κ_0, du_0, dv_0 satisfy

$$\begin{aligned}(L-\kappa_0 E)\,du_0 + (M-\kappa_0 F)\,dv_0 &= 0\\ (M-\kappa_0 F)\,du_0 + (N-\kappa_0 G)\,dv_0 &= 0\end{aligned} \qquad (a)$$

Suppose κ_0 is a principal curvature with associated principal direction $du_0 : dv_0$. Recall that the principal curvatures are the maximum or minimum values of the normal curvature κ_n. Thus if

$$\kappa_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

takes on a maximum or minimum κ_0 at (du_0, dv_0), then, from calculus, the partial derivatives

$$\left.\frac{\partial\kappa_n}{\partial\,du}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\frac{\partial\kappa_n}{\partial\,dv}\right|_{(du_0,dv_0)} = 0$$

or, differentiating,

$$\left.\frac{\mathrm{I}\,\mathrm{II}_{du} - \mathrm{II}\,\mathrm{I}_{du}}{\mathrm{I}^2}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\frac{\mathrm{I}\,\mathrm{II}_{dv} - \mathrm{II}\,\mathrm{I}_{dv}}{\mathrm{I}^2}\right|_{(du_0,dv_0)} = 0$$

Multiplying by I gives

$$\left.\mathrm{II}_{du} - \frac{\mathrm{II}}{\mathrm{I}}\mathrm{I}_{du}\right|_{(du_0,dv_0)} = 0 \quad\text{and}\quad \left.\mathrm{II}_{dv} - \frac{\mathrm{II}}{\mathrm{I}}\mathrm{I}_{dv}\right|_{(du_0,dv_0)} = 0$$

But $(\mathrm{II}/\mathrm{I})|_{(du_0,dv_0)} = \kappa_n|_{(du_0,dv_0)} = \kappa_0$. Hence

$$(\mathrm{II}_{du} - \kappa_0\mathrm{I}_{du})|_{(du_0,dv_0)} = 0 \quad\text{and}\quad (\mathrm{II}_{dv} - \kappa_0\mathrm{I}_{dv})|_{(du_0,dv_0)} = 0$$

Since $\mathrm{II}_{du} = 2L\,du + 2M\,dv$ and $\mathrm{I}_{du} = 2E\,du + 2F\,dv$, etc.,

$$\begin{aligned}(L\,du_0 + M\,dv_0) - \kappa_0(E\,du_0 + F\,dv_0) &= 0\\ (M\,du_0 + N\,dv_0) - \kappa_0(F\,du_0 + G\,dv_0) &= 0\end{aligned}$$

which gives the required result. Now conversely, suppose that κ_0, du_0, dv_0, $du_0^2 + dv_0^2 \neq 0$, satisfy equation (*a*) above. Then κ_0 together with the principal curvatures must satisfy

$$\det\begin{pmatrix} L-\kappa E & M-\kappa F \\ M-\kappa F & N-\kappa G \end{pmatrix} = 0$$

or, expanding,

$$(EG-F^2)\kappa^2 - (EN+GL-2FM)\kappa + (LN-M^2) = 0 \tag{b}$$

Now suppose P is an umbilical point with curvature κ. Since κ is taken in every direction, the coefficients of (*a*) must all be zero, i.e. $\kappa = E/L = M/F = N/G$. But then it follows from Problem 9.14 that equation (*b*) has a single root with multiplicity two and therefore $\kappa = \kappa_0$. Thus κ_0 is the principal curvature, and every direction including $du_0 : dv_0$ is a principal direction. If P is a nonumbilical point, κ_0 must be one of the two distinct roots of (*b*), i.e. one of the two principal curvatures at a nonumbilical point with principal direction $du_0 : dv_0$, which proves the theorem.

9.17. Show that the curvature κ at a point P on the curve C of intersection of two surfaces satisfies

$$\kappa^2 \sin^2\alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\cos\alpha$$

where κ_1 and κ_2 are the normal curvatures of the surfaces in the direction of C at P, and α is the angle between the normals to the surfaces at P.

From equation (*9.15*), page 179,

$$\kappa_1\mathbf{N}_2 = \kappa(\mathbf{n}\cdot\mathbf{N}_1)\mathbf{N}_2 \quad\text{and}\quad \kappa_2\mathbf{N}_1 = \kappa(\mathbf{n}\cdot\mathbf{N}_2)\mathbf{N}_1$$

Subtracting and using the vector identity of Theorem 1.8, page 10,

$$\kappa_1\mathbf{N}_2 - \kappa_2\mathbf{N}_1 = \kappa[(\mathbf{n}\cdot\mathbf{N}_1)\mathbf{N}_2 - (\mathbf{n}\cdot\mathbf{N}_2)\mathbf{N}_1] = \kappa(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}$$

It follows further, using the identity $[\mathbf{F}_1]$, page 10, that

$$\begin{aligned}(\kappa_1\mathbf{N}_2-\kappa_2\mathbf{N}_1)\cdot(\kappa_1\mathbf{N}_2-\kappa_2\mathbf{N}_1) &= \kappa^2[(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}]\cdot[(\mathbf{N}_1\times\mathbf{N}_2)\times\mathbf{n}] \\ &= \kappa^2[(\mathbf{N}_1\times\mathbf{N}_2)\cdot(\mathbf{N}_1\times\mathbf{N}_2) - ((\mathbf{N}_1\times\mathbf{N}_2)\cdot\mathbf{n})^2] \\ &= \kappa^2(\mathbf{N}_1\times\mathbf{N}_2)\cdot(\mathbf{N}_1\times\mathbf{N}_2)\end{aligned}$$

where we used the fact that $\mathbf{N}_1\times\mathbf{N}_2$ is a vector parallel to the tangent to the curve and hence $(\mathbf{N}_1\times\mathbf{N}_2)\cdot\mathbf{n} = 0$. Expanding,

$$\kappa_1^2(\mathbf{N}_2\cdot\mathbf{N}_2) - 2\kappa_1\kappa_2(\mathbf{N}_1\cdot\mathbf{N}_2) + \kappa_2^2(\mathbf{N}_1\cdot\mathbf{N}_1) = \kappa^2|\mathbf{N}_1\times\mathbf{N}_2|^2$$

or

$$\kappa_1^2 - 2\kappa_1\kappa_2\cos\alpha + \kappa_2^2 = \kappa^2\sin^2\alpha$$

9.18. Prove that at each point on a patch,

$$\mathbf{N}_u\times\mathbf{N}_v = K(\mathbf{x}_u\times\mathbf{x}_v)$$

where K is the Gaussian curvature at the point.

Since $\mathbf{N}$ is a function of unit length, $\mathbf{N}_u$ and $\mathbf{N}_v$ are orthogonal to $\mathbf{N}$ and hence parallel to the tangent plane. It follows that we can write $\mathbf{N}_u = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{N}_v = c\mathbf{x}_u + d\mathbf{x}_v$ where a, b, c, d are to be determined. Note that

$$\mathbf{N}_u\times\mathbf{N}_v = (a\mathbf{x}_u + b\mathbf{x}_v)\times(c\mathbf{x}_u + d\mathbf{x}_v) = (ad-bc)(\mathbf{x}_u\times\mathbf{x}_v)$$

Thus it remains to show that

$$ad - cb = \det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = K$$

From the above and (*9.10*), page 175, we have

$$\mathbf{x}_u\cdot\mathbf{N}_u = a\mathbf{x}_u\cdot\mathbf{x}_u + b\mathbf{x}_u\cdot\mathbf{x}_v = aE + bF = -L$$

Similarly,
$$\mathbf{x}_v \cdot \mathbf{N}_u = aF + bG = -M$$
$$\mathbf{x}_u \cdot \mathbf{N}_v = cE + dF = -M$$
$$\mathbf{x}_v \cdot \mathbf{N}_v = cF + dG = -N$$

These equations can be written as the matrix product
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

Hence
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \det\begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

or
$$ad - cb = \frac{LN - M^2}{EG - F^2} = K$$

which completes the proof.

LINES OF CURVATURE

9.19. Determine the principal directions to $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2+v^2)\mathbf{e}_3$ at $u = 1,\ v = 1$ and verify Rodrigues' formula in each direction.

From Example 9.11, page 185, $E = 1+4u^2$, $F = 4uv$, $G = 1+4v^2$, $L = 2(4u^2+4v^2+1)^{-1/2}$, $M = 0$, $N = 2(4u^2+4v^2+1)^{-1/2}$. At $u = 1$, $v = 1$, we have $E = 5$, $F = 4$, $G = 5$, $L = 2/3$, $M = 0$, $N = 2/3$. From equation (9.25), page 185, the principal directions at $u = 1$, $v = 1$ are the solutions of
$$-\tfrac{8}{3}\,du^2 + \tfrac{8}{3}\,dv^2 = 0 \quad\text{or}\quad (du+dv)(du-dv) = 0$$

Hence $du_1 : dv_1 = 1 : -1$ and $du_2 : dv_2 = 1 : 1$. We have further,
$$\mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2v\mathbf{e}_3$$
$$\mathbf{N} = (4u^2+4v^2+1)^{-1/2}(-2u\mathbf{e}_1 - 2v\mathbf{e}_2 + \mathbf{e}_3)$$
$$\mathbf{N}_u = (4u^2+4v^2+1)^{-3/2}[-(8v^2+2)\mathbf{e}_1 + 8uv\mathbf{e}_2 - 4u\mathbf{e}_3]$$
$$\mathbf{N}_v = (4u^2+4v^2+1)^{-3/2}[8uv\mathbf{e}_1 - (8u^2+2)\mathbf{e}_2 - 4v\mathbf{e}_3]$$

At $u = 1$, $v = 1$,
$$\mathbf{x}_u = \mathbf{e}_1 + 2\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2\mathbf{e}_3, \quad \mathbf{N}_u = \tfrac{1}{27}(-10\mathbf{e}_1 + 8\mathbf{e}_2 - 4\mathbf{e}_3), \quad \mathbf{N}_v = \tfrac{1}{27}(8\mathbf{e}_1 - 10\mathbf{e}_2 - 4\mathbf{e}_3)$$
$$d\mathbf{N}_1 = \mathbf{N}_u\,du_1 + \mathbf{N}_v\,dv_1 = \tfrac{1}{27}(-18\mathbf{e}_1 + 18\mathbf{e}_2) = -\tfrac{18}{27}(\mathbf{e}_1 - \mathbf{e}_2), \quad d\mathbf{x}_1 = \mathbf{x}_u\,du_1 + \mathbf{x}_v\,dv_1 = \mathbf{e}_1 - \mathbf{e}_2$$

Thus $d\mathbf{N}_1 = -\frac{18}{27}\,d\mathbf{x}_1$. Also,
$$d\mathbf{N}_2 = \mathbf{N}_u\,du_2 + \mathbf{N}_v\,dv_2 = \tfrac{1}{27}[-2\mathbf{e}_1 - 2\mathbf{e}_2 - 8\mathbf{e}_3] = -\tfrac{2}{27}[\mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3]$$
$$d\mathbf{x}_2 = \mathbf{x}_u\,du_2 + \mathbf{x}_v\,dv_2 = \mathbf{e}_1 + \mathbf{e}_2 + 4\mathbf{e}_3$$

Hence $d\mathbf{N}_2 = -\frac{2}{27}\,d\mathbf{x}_2$, verifying Rodrigues' formula.

9.20. Prove that the solutions to $A\,du^2 + 2B\,du\,dv + G\,dv^2 = 0$ form orthogonal families of curves on a patch if and only if $EC - 2FB + GA = 0$.

Suppose
$$A\,du^2 + 2B\,du\,dv + C^2\,dv^2 = (A'\,du + B'\,dv)(C'\,du + D'\,dv)$$

Then one of the family of curves is the solution to $A'\,du + B'\,dv = 0$ and the other is the solution to, say, $C'\,\delta u + D'\,\delta v = 0$. Along the first, $du : dv = B' : -A'$; and along the second, $\delta u : \delta v = D' : -C'$. But from Theorem 9.1, page 173, the families are orthogonal if and only if
$$E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = EB'D' - F(B'C' + D'A') + GA'C'$$
$$= EC - 2FB + GA = 0$$

9.21. Prove Theorem 9.11. That is, prove that if P is a point on a surface of class $\geqq 2$, then there exists a coordinate patch containing P such that the directions of the parameter lines are principal directions.

It is sufficient to consider the case where P is a nonumbilical point. For otherwise every direction is a principal direction and any patch containing P will suffice. Now suppose $\mathbf{x} = \mathbf{x}(u, v)$ is an arbitrary patch containing the nonumbilical point P and suppose $du_1 : dv_1$ and $du_2 : dv_2$ are the principal directions at P. Consider the linear parameter transformation $u = du_1\,\theta + du_2\,\phi$, $v = dv_1\,\theta + dv_2\,\phi$. Note that $\dfrac{\partial(u,v)}{\partial(\theta,\phi)} = \det\begin{pmatrix} du_1 & du_2 \\ dv_1 & dv_2 \end{pmatrix} \neq 0$, since the directions $du_1 : dv_1$ and $du_2 : dv_2$ are distinct. Thus the parameter transformation is an allowable transformation of class C^∞. It follows that $\mathbf{x} = \mathbf{x}^*(\theta,\phi) = \mathbf{x}(u(\theta,\phi), v(\theta,\phi))$ is a patch of class C^2 containing P and that

$$\mathbf{x}_\theta \;=\; \mathbf{x}_u(\partial u/\partial\theta) + \mathbf{x}_v(\partial v/\partial\theta) \;=\; \mathbf{x}_u\,du_1 + \mathbf{x}_v\,dv_1$$

$$\mathbf{x}_\phi \;=\; \mathbf{x}_u(\partial u/\partial\phi) + \mathbf{x}_v(\partial u/\partial\phi) \;=\; \mathbf{x}_u\,du_2 + \mathbf{x}_v\,dv_2$$

Namely the θ and ϕ parameter curves at P are in the direction of the principal directions.

9.22. *Euler's Theorem.* Prove that the normal curvature at a point on a surface of class $\geqq 2$ in the direction of a tangent line L is given by

$$\kappa_n \;=\; \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

where κ_1 and κ_2 are the principal curvatures at P and α is the angle between L and a tangent line in the principal direction corresponding to κ_1.

The theorem is clearly true if P is an umbilical point where $\kappa_1 = \kappa_2 = \kappa_n$. Otherwise let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch containing P such that the u- and v-parameter curves are in the direction of the principal directions. Then from Theorem 9.12, page 186, $F = M = 0$ and the normal curvature in any direction $du : dv$ is $\kappa_n = \dfrac{L\,du^2 + N\,dv^2}{E\,du^2 + G\,dv^2}$. From Theorem 9.13 it follows that the principal curvatures are $\kappa_1 = L/E$ and $\kappa_2 = N/G$ respectively. Substituting,

$$\kappa_n \;=\; \kappa_1 \frac{E\,du^2}{E\,du^2 + G\,dv^2} + \kappa_2 \frac{G\,dv^2}{E\,du^2 + G\,dv^2}$$

Now if α and β are the angles between an arbitrary tangent line L with direction numbers $du : dv$ and the tangent lines in the principal directions $1:0$ and $0:1$ respectively, we have from equation (*9.6*), page 173,

$$\cos\alpha \;=\; \frac{E\,du}{\sqrt{E\,du^2 + G\,dv^2}\,\sqrt{E}} \qquad \text{and} \qquad \cos\beta \;=\; \frac{G\,dv^2}{\sqrt{E\,du^2 + G\,dv}\,\sqrt{G}}$$

Squaring and substituting,

$$\kappa_n \;=\; \kappa_1 \cos_2^2\alpha + \kappa_2 \cos^2\beta$$

But the principal directions are perpendicular, i.e. $\beta = \pi/2 - \alpha$. Hence $\kappa_n = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$.

9.23. Complete the proof of Theorem 9.14. That is, prove that $du : dv$ is a principal direction at a point on a patch if for some scalar κ

$$d\mathbf{N} \;=\; -\kappa\, d\mathbf{x}$$

in the direction $du : dv$.

From $d\mathbf{N} = -\kappa\, d\mathbf{x}$ we have $(d\mathbf{N} + \kappa\, d\mathbf{x}) \cdot \mathbf{x}_u = 0$ and $(d\mathbf{N} + \kappa\, d\mathbf{x}) \cdot \mathbf{x}_v = 0$ or

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_u \;=\; 0$$

$$[(\mathbf{N}_u\,du + \mathbf{N}_v\,dv) + \kappa(\mathbf{x}_u\,du + \mathbf{x}_v\,dv)] \cdot \mathbf{x}_v \;=\; 0$$

or

$$(-\mathbf{N}_u \cdot \mathbf{x}_u - \kappa\mathbf{x}_u \cdot \mathbf{x}_u)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_u - \kappa\mathbf{x}_v \cdot \mathbf{x}_u)\,dv \;=\; 0$$

$$(-\mathbf{N}_u \cdot \mathbf{x}_v - \kappa\mathbf{x}_u \cdot \mathbf{x}_v)\,du + (-\mathbf{N}_v \cdot \mathbf{x}_v - \kappa\mathbf{x}_v \cdot \mathbf{x}_v)\,dv \;=\; 0$$

or

$$(L-\kappa E)\,du + (M-\kappa F)\,dv = 0$$
$$(M-\kappa F)\,du + (N-\kappa G)\,dv = 0$$

From Theorem 9.5, page 183, it follows that κ is a principal curvature and $du:dv$ is the corresponding principal direction.

9.24. Prove Theorem 9.9. That is, prove that $du:dv$ is a principal direction at a point on a patch if and only if du and dv satisfy

$$(EM-LF)\,du^2 + (EN-LG)\,du\,dv + (FN-MG)\,dv^2 = 0$$

From Theorem 9.5, $du:dv$ is a principal direction if and only if for some κ,

$$(L-\kappa E)\,du + (M-\kappa F)\,dv = 0$$
$$(M-\kappa F)\,du + (N-\kappa G)\,dv = 0$$

or

$$(L\,du + M\,dv) - \kappa(E\,du + F\,dv) = 0$$
$$(M\,du + N\,dv) - \kappa(F\,du + G\,dv) = 0$$

But the above can have a nontrivial solution $(1, -\kappa)$ if and only if

$$\det\begin{pmatrix} L\,du + M\,dv & E\,du + F\,dv \\ M\,du + N\,dv & F\,du + G\,dv \end{pmatrix} = 0$$

or, expanding,

$$(EM-LF)\,du^2 + (EN-LG)\,du\,dv + (FN-MG)\,dv^2 = 0$$

9.25. If two surfaces intersect at a constant angle and if the curve of intersection is a line of curvature on one of the surfaces, prove that it is a line of curvature on the other surface.

Since the surfaces, say M_1 and M_2, intersect at a constant angle, along the curve of intersection $N_1 \cdot N_2 =$ constant. Hence

$$0 = \frac{d}{dt}(\mathbf{N}_1 \cdot \mathbf{N}_2) = \left(\frac{d}{dt}\mathbf{N}_1\right) \cdot \mathbf{N}_2 + \mathbf{N}_1 \cdot \frac{d}{dt}\mathbf{N}_2$$

Assuming the curve of intersection is a line of curvature along M_1, then (Rodrigues' formula) $\frac{d\mathbf{N}_1}{dt} = -\kappa_1 \frac{d\mathbf{x}}{dt}$. Hence

$$-\kappa_1 \frac{d\mathbf{x}}{dt} \cdot \mathbf{N}_2 + \mathbf{N}_1 \cdot \frac{d\mathbf{N}_2}{dt} = 0$$

But $d\mathbf{x}/dt$ is orthogonal to $\mathbf{N}_2$, i.e. $(d\mathbf{x}/dt) \cdot \mathbf{N}_2 = 0$. Thus $\mathbf{N}_1 \cdot (d\mathbf{N}_2/dt) = 0$. Hence $d\mathbf{N}_2/dt$ is orthogonal to $\mathbf{N}_1$. But $d\mathbf{N}_2/dt$ is also orthogonal to $\mathbf{N}_2$, since $\mathbf{N}_2$ is of unit length. It follows that $d\mathbf{N}_2/dt$ is parallel to $d\mathbf{x}/dt$. Namely, there exists κ_2 such that $d\mathbf{N}_2/dt = -\kappa_2(d\mathbf{x}/dt)$. Hence the curve of intersection is also a line of curvature along M_2.

9.26. *Third Fundamental Form.* The third fundamental form is defined by $\mathrm{III} = d\mathbf{N} \cdot d\mathbf{N}$. Prove that $\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$ where H and K are the mean and Gaussian curvatures respectively.

It is easily verified that III is invariant, in the same sense as I. Note that II and H change sign with a change in orientation. Thus it is sufficient to consider a fixed point P and a coordinate patch containing P such that the directions of the u- and v-parameter curves at P are principal directions. From Rodrigues' formula,

$$\mathbf{N}_u = -\kappa_1 \mathbf{x}_u \quad \text{and} \quad \mathbf{N}_v = -\kappa_2 \mathbf{x}_v$$

where κ_1 and κ_2 are the principal curvatures. It follows that for arbitrary (du, dv),

$$d\mathbf{N} = \mathbf{N}_u\,du + \mathbf{N}_v\,dv = -\kappa_1\mathbf{x}_u\,du - \kappa_2\mathbf{x}_v\,dv$$
$$= -\kappa_1\mathbf{x}_u\,du - \kappa_1\mathbf{x}_v\,dv + \kappa_1\mathbf{x}_v\,dv - \kappa_2\mathbf{x}_v\,dv = -\kappa_1\,d\mathbf{x} + (\kappa_1 - \kappa_2)\mathbf{x}_v\,dv$$

or

$$d\mathbf{N} + \kappa_1\,d\mathbf{x} = (\kappa_1 - \kappa_2)\mathbf{x}_v\,dv$$

Also, $$d\mathbf{N} = -\kappa_1\mathbf{x}_u\,du + \kappa_2\mathbf{x}_u\,du - \kappa_2\mathbf{x}_u\,du - \kappa_2\mathbf{x}_v\,dv = -\kappa_2\,d\mathbf{x} + (\kappa_2-\kappa_1)\mathbf{x}_u\,du$$

or $$d\mathbf{N} + \kappa_2\,d\mathbf{x} = (\kappa_2-\kappa_1)\mathbf{x}_u\,du$$

Thus $$(d\mathbf{N}+\kappa_1\,d\mathbf{x})\cdot(d\mathbf{N}+\kappa_2\,d\mathbf{x}) = (\kappa_1-\kappa_2)(\kappa_2-\kappa_1)\,du\,dv\,\mathbf{x}_u\cdot\mathbf{x}_v$$

Since the lines of curvature are orthogonal at P, we have $\mathbf{x}_u\cdot\mathbf{x}_v = 0$. Then

$$(d\mathbf{N}+\kappa_1\,d\mathbf{x})\cdot(d\mathbf{N}+\kappa_2\,d\mathbf{x}) = 0$$

or $$d\mathbf{N}\cdot d\mathbf{N} + (\kappa_1+\kappa_2)\,d\mathbf{N}\cdot d\mathbf{x} + \kappa_1\kappa_2\,d\mathbf{x}\cdot d\mathbf{x} = 0$$

Hence $\mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0$.

9.27. Show that the principal directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions to

$$\det\begin{pmatrix} dx_1 & f_{x_1} & df_{x_1} \\ dx_2 & f_{x_2} & df_{x_2} \\ dx_3 & f_{x_3} & df_{x_3}\end{pmatrix} = 0, \qquad f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 = 0$$

If $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is tangent, then $f_{x_1}dx_1 + f_{x_2}dx_2 + f_{x_3}dx_3 = 0$. Hence $\mathbf{G} = f_{x_1}\mathbf{e}_1 + f_{x_2}\mathbf{e}_2 + f_{x_3}\mathbf{e}_3$ is normal, and $\mathbf{N} = \mathbf{G}/|\mathbf{G}|$. Thus

$$d\mathbf{N} = \frac{d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}$$

where $d\mathbf{G} = df_{x_1}\,\mathbf{e}_1 + df_{x_2}\,\mathbf{e}_2 + df_{x_3}\mathbf{e}_3$. Now suppose $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is in a principal direction. Then from Rodrigues' formula, $d\mathbf{x}$ is parallel to $d\mathbf{N}$. Hence the three vectors $d\mathbf{x}$, $\mathbf{N}$ and $d\mathbf{N}$ are dependent, or

$$0 = [d\mathbf{x}\,\mathbf{N}\,d\mathbf{N}] = \left[d\mathbf{x}\,\frac{\mathbf{G}}{|\mathbf{G}|}\left(\frac{d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}\right)\right] = \frac{1}{|\mathbf{G}|^2}[d\mathbf{x}\,\mathbf{G}\,d\mathbf{G}] - \frac{\mathbf{G}\cdot d\mathbf{G}}{|\mathbf{G}|^4}[d\mathbf{x}\,\mathbf{G}\mathbf{G}]$$

But $[d\mathbf{x}\,\mathbf{G}\mathbf{G}] = 0$; hence $[d\mathbf{x}\,\mathbf{G}\,d\mathbf{G}] = 0$. The required result follows.

ASYMPTOTIC LINES — CONJUGATE FAMILIES OF CURVES

9.28. Prove that the parameter curves on the surface $\mathbf{x} = \mathbf{x}_1(u) + \mathbf{x}_2(v)$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are arbitrary, are conjugate families of curves.

$\mathbf{x}_u = \mathbf{x}_1'(u)$, $\mathbf{x}_v = \mathbf{x}_2'(v)$, $\mathbf{x}_{uv} \equiv 0$. Hence $M = \mathbf{x}_{uv}\cdot\mathbf{N} \equiv 0$. Thus from Theorem 9.20, page 189, the parameter curves are conjugate families.

9.29. Prove Theorem 9.20; that is, prove that at an elliptic or hyperbolic point every direction has a unique conjugate direction.

From equation (9.29), page 188, $\delta u:\delta v$ is conjugate to $du:dv$ if and only if

$$(L\,du + M\,dv)\,\delta u + (M\,du + N\,dv)\,\delta v = 0$$

The above equation has a unique solution $\delta u:\delta v$, $\delta u^2 + \delta v^2 \neq 0$, if and only if the coefficients do not both vanish. Thus, given $du:dv$, there exists a unique $\delta u:\delta v$ if and only if

$$(L\,du + M\,dv)^2 + (M\,du + N\,dv)^2 \neq 0$$

or $$(L^2+M^2)\,du^2 + 2(LM+MN)\,du\,dv + (M^2+N^2)\,dv^2 \neq 0$$

But this equation is different from zero for all $du:dv$, $du^2+dv^2 \neq 0$, if and only if its discriminant

$$(L^2+M^2)(M^2+N^2) - (LM+MN)^2 > 0$$

or, expanding, $(LN-M^2)^2 > 0$ or $LN-M^2 \neq 0$. Thus every direction $du:dv$ has a unique conjugate direction, $\delta u:\delta v$ if and only if $LN-M^2 \neq 0$, which proves the theorem and in fact also its converse.

9.30. Prove Theorem 9.18; that is, prove that the torsion along an asymptotic line which is not a straight line satisfies $\tau^2 = -K$.

From Theorem 9.17, page 188, the osculating plane at each point on the asymptotic line is tangent to the surface. Hence along the curve the binormal $\mathbf{b} = \pm\mathbf{N}$. It follows that $\mathbf{b} = \pm(d\mathbf{N}/ds) = -\tau\mathbf{n}$ and $(d\mathbf{N}/ds)\cdot(d\mathbf{N}/ds) = \tau^2(\mathbf{n}\cdot\mathbf{n}) = \tau^2$. From Problem 9.26, $\text{III} - 2H\,\text{II} + K\,\text{I} = 0$. But along an asymptotic line $\text{II} = 0$, $\text{III} = (d\mathbf{N}/ds)\cdot(d\mathbf{N}/ds) = \tau^2$ and $\text{I} = (d\mathbf{x}/ds)\cdot(d\mathbf{x}/ds) = 1$. Hence $\tau^2 + K = 0$ or $\tau^2 = -K$.

9.31. Show that the asymptotic directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions of

$$dx_1\,df_{x_1} + dx_2\,df_{x_2} + dx_3\,df_{x_3} = 0, \qquad f_{x_1}\,dx_1 + f_{x_2}\,dx_2 + f_{x_3}\,dx_3 = 0$$

From Problem 9.27, $\mathbf{G} = f_{x_1}\mathbf{e}_1 + f_{x_2}\mathbf{e}_2 + f_{x_3}\mathbf{e}_3$ is normal to the surface. $\mathbf{N} = \dfrac{\mathbf{G}}{|\mathbf{G}|}$ and $d\mathbf{N} = \dfrac{d\mathbf{G}}{|\mathbf{G}|} - \dfrac{(\mathbf{G}\cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}$. Hence if $d\mathbf{x} = dx_1\,\mathbf{e}_1 + dx_2\,\mathbf{e}_2 + dx_3\,\mathbf{e}_3$ is in an asymptotic direction,

$$0 = \text{II} = -d\mathbf{x}\cdot d\mathbf{N} = -\frac{d\mathbf{x}\cdot d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G}\cdot d\mathbf{G})(d\mathbf{x}\cdot\mathbf{G})}{|\mathbf{G}|^3}$$

But $d\mathbf{x}\cdot\mathbf{G} = 0$. Thus $d\mathbf{x}\cdot d\mathbf{G} = 0$, or $dx_1\,df_{x_1} + dx_2\,df_{x_2} + dx_3\,df_{x_3} = 0$.

9.32. Use the above to find the asymptotic curves on the surface $f = x_3 - x_1 \sin x_2 = 0$, $-\pi/2 < x_2 < \pi/2$, $x_1 > 0$.

$f_{x_1} = -\sin x_2$, $f_{x_2} = -x_1 \cos x_2$, $f_{x_3} = 1$, $df_{x_1} = -\cos x_2\,dx_2$, $df_{x_2} = -\cos x_2\,dx_1 + x_1 \sin x_2\,dx_2$, $df_{x_3} = 0$. The equation

$$df_{x_1}\,dx_1 + df_{x_2}\,dx_2 + df_{x_3}\,dx_3 = -2\cos x_2\,dx_1\,dx_2 + x_1 \sin x_2\,dx_2^2 = 0$$

has factors $dx_2 = 0$ and $-2\cos x_2\,dx_1 + x_1 \sin x_2\,dx_2 = 0$ which have solutions $x_2 = C$ and $x_1 = K\sec^{1/2} x_2$. Substituting the first into $x_3 - x_1 \sin x_2 = 0$ gives the family of straight lines $x_1 = t$, $x_2 = C$, $x_3 = t \sin C$, $t > 0$. The second gives the family of curves $x_1 = K\sec^{1/2} u$, $x_2 = u$, $x_3 = K\sec^{1/2} u \sin u$, $-\pi/2 < u < \pi/2$.

Supplementary Problems

9.33. Show that the parameter curves on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\,\mathbf{e}_3$ are an orthogonal family of curves if and only if $f_u f_v \equiv 0$.

9.34. Show that the first fundamental form on the tangent surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{t}(s)$ to the curve $\mathbf{y} = \mathbf{y}(s)$ is $\text{I} = (1 + u^2\kappa^2)\,ds^2 + ds\,du + du^2$.

9.35. Show that the first fundamental form of the surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{b}(s)$ generated by the binormal $\mathbf{b}(s)$ of a curve $\mathbf{y} = \mathbf{y}(s)$ is $\text{I} = (1 + u^2\tau^2)\,ds^2 + du^2$.

9.36. Find the length of the arc $\theta = \displaystyle\int_{\pi/4}^{t} \frac{1}{\sin\tau}\,d\tau$, $\phi = t$, $\pi/4 \le t \le \pi/2$, on the sphere $\mathbf{x} = (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_1 + (\cos\phi)\mathbf{e}_3$. *Ans.* $\sqrt{2}\,\pi/4$

9.37. Show that the surface area on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is given by the integral $A = \displaystyle\iint_W \sqrt{1 + f_u^2 + f_v^2}\,du\,dv$.

9.38. Prove that on the intersection of two coordinate patches $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ on a surface, $EG - F^2 = (E^*G^* - F^{*2})\left[\dfrac{\partial(\theta,\phi)}{\partial(u,v)}\right]^2$.

9.39. Show that the curves on the surface $\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + (a\theta+b)\mathbf{e}_3$ satisfying $(u^2+a^2)\,d\theta^2 - du^2 = 0$ are orthogonal families of curves.

9.40. Show that the θ-parameter curves on the surface $\mathbf{x} = (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + f(\theta)\mathbf{e}_3$ are parallel.

9.41. Show that the second fundamental form on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is

$$\text{II} = (f_u^2 + f_v^2 + 1)^{-1/2}[f_{uu}\,du^2 + 2f_{uv}\,du\,dv + f_{vv}\,dv^2]$$

9.42. Show that all points on the general cylinder $\mathbf{x} = \mathbf{y}(s) + u\mathbf{g}$, $\mathbf{g} = \text{constant}$, are parabolic or planar.

9.43. If $\mathbf{x} = \mathbf{x}(u,v)$, $\mathbf{x} = \mathbf{x}^*(\theta,\phi)$ are patches on a surface such that on the intersection $\partial(\theta,\phi)/\partial(u,v) > 0$, show that the second fundamental coefficients transform as follows:

$$L = L^*\theta_u^2 + 2M^*\theta_u\phi_u + M^*\phi_u^2$$
$$M = L^*\theta_u\theta_v + M^*(\theta_u\phi_v + \phi_u\theta_v) + N^*\phi_u\phi_v$$
$$N = L^*\theta_v^2 + 2M^*\theta_v\phi_v + N^*\phi_v^2$$

9.44. Show that the Gauss and mean curvatures on $\mathbf{x} = (u+v)\mathbf{e}_1 + (u-v)\mathbf{e}_2 + uv\mathbf{e}_3$ at $u=1$, $v=1$ are $K = 1/16$ and $H = 1/8\sqrt{2}$.

9.45. Show that the mean curvature is zero at every point on the surface of revolution

$$\mathbf{x} = (\cosh u\cos\theta)\mathbf{e}_1 + (\cosh u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$$

9.46. Show that the principal curvatures of the surface $x_1\sin x_3 - x_2\cos x_3 = 0$ are $\pm 1/(x_1^2 + x_2^2 + 1)$.

9.47. Show that the lines of curvature on the surface $\mathbf{x} = (u\cos\theta)\mathbf{e}_1 + (u\sin\theta)\mathbf{e}_2 + u\mathbf{e}_3$ are the images of $\log(u + \sqrt{u^2+1}) - v = C$ and $\log(u + \sqrt{u^2+1}) + v = K$.

9.48. Find the principal curvatures and principal directions on

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (4u^2+v^2)\mathbf{e}_3$$

at $u=0$, $v=0$ using Dupin's indicatrix.

9.49. Prove that the family of curves on a patch which are orthogonal to the family of curves given by $A\,du + B\,dv = 0$ are given by the solutions to $(EB - FA)\,du + (FB - GA)\,dv = 0$.

9.50. The parameter curves on the surface

$$\mathbf{x} = e^{(u-v)/2}\left(\cos\frac{u+v}{2}\right)\mathbf{e}_1 + e^{(u-v)/2}\left(\sin\frac{u+v}{2}\right)\mathbf{e}_2 + \left(\frac{u-v}{2}\right)\mathbf{e}_3$$

are asymptotic lines. Verify that along the u-parameter curve $v = 0$, the torsion satisfies $\tau^2 = -K$.

9.51. Show that the asymptotic lines on the surface $x_3 - x_1^4 + x_2^4 = 0$ are the intersections of the surface and the families of cylinders $x_1^2 + x_2^2 = C$ and $x_1^2 - x_2^2 = K$.

9.52. Prove that the directions of curvature bisect the asymptotic directions.

9.53. Show that the mean curvature is zero on a surface whose asymptotic lines are orthogonal families of curves.

9.54. If a sphere or a plane intersects a surface at a constant angle, prove that curve of intersection is a line of curvature.

9.55. Prove that the sum of the normal curvatures at a point on a surface in any pair of orthogonal directions is constant.

9.56. If a surface has a one-parameter family of plane asymptotic curves other than straight lines, prove that the surface is a plane.

9.57. Suppose R is a region on a patch on a surface. The endpoints of the unit normals in R form a set R' on the unit sphere called the spherical image of R. Show that the ratio of the area of R' to the area of R tends to $|K|$ at a point P when R shrinks down to the point P. *Hint.* Problem 9.18, page 194.

9.58. If $K \neq 0$ at a point P on a surface, show that there is a neighborhood of P in which the points can be put into a 1-1 correspondence with the spherical image of the neighborhood (see Problem 9.57).