Your Name:

Final Exam
Fall 2021, Complex Analysis II
Mathematics Education, Chungbuk National University
15.12.2021 15:00–16:40

Instructions: Please write your name on each page. If you want some portion of your writings on your answer sheet not to be graded, just cross it out. You are not allowed to use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

Extra sheets: Use the blank page on the back of each answer sheet as your scrap paper. Your work on blank pages will not be graded. Do not write your answers on those blank pages. If you need more space for writing down your answers, please ask for additional sheets.

1. Answer whether each of the following statements is true or false. No need to give reasons or details. **Just say true or false**. 2 points for each correct answer, 0 point for no answer, and -2 points for each incorrect answer.

(1) Any complex number can be obtained as an image of the function $z \mapsto e^{1/z}$ defined on the punctured unit disc $\mathbb{D} - \{0\}$. False

(2) The totality of Möbius transformations endowed with composition of mappings is isomorphic as groups to the group of all invertible 2×2 matrices with entries in $\mathbb{C}$ endowed with matrix multiplication. False

(3) A Möbius transformation is completely determined by its values at three different points on the extended complex plane. True

(4)
$$\int_1^2 \frac{1}{(x-1)(2-x)(5-x)(9-x)} dx = \int_5^9 \frac{1}{(x-1)(x-2)(x-5)(9-x)} dx$$ True

(5) The Riemann mapping states that every simply connected domain in the complex plane is biholomorphic to the unit disc. False

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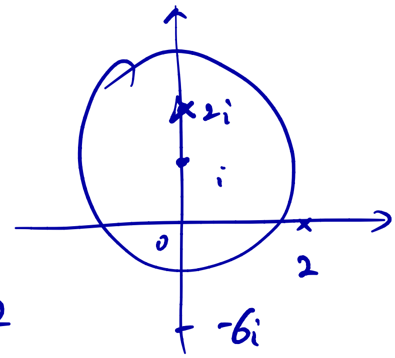
2. In the complex plane, let C be a circle traversing in counter-clockwise direction with center at i and the radius 2. Compute the following line integral

$$\int_C \left\{ \frac{4e^{-iz}}{(z+6i)(z-2i)} + \overline{z} \right\} dz$$

(Here $\overline{z}$ denotes the complex conjugate of z.) [15 points]



$$\int_C \left\{ \frac{4e^{-iz}}{(z+6i)(z-2i)} + \overline{z} \right\} dz$$

$|z-2| = 2$

$(\overline{z}-2) = \frac{4}{z-2}$

$$= \int_C \left\{ \frac{4e^{-iz}}{(z+6i)(z-2i)} + 2 + \frac{4}{z-2} \right\} dz$$

$$= 2\pi i \operatorname{Res}\left(\frac{4e^{-iz}}{(z+6i)(z-2i)} ; 2i \right) + 2\pi i \operatorname{Res}\left(\frac{4}{z-2} ; 2 \right)$$

$$= 2\pi i \frac{4e^{2}}{8i} + 8\pi i = (e^{2}+8i)\pi$$ ✓

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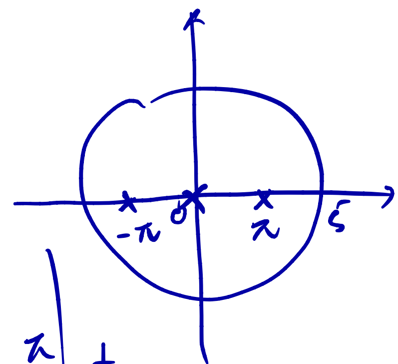
3. In the complex plane $\mathbb{C}$, a curve $C = \{z \in \mathbb{C} | |z| = 5\}$ is making one cycle in counter-clockwise direction. Compute the value of

$$\frac{1}{2\pi i}\int_C \frac{\cos z}{\sin z} dz.$$

[10 points]



$$\frac{1}{2\pi i}\int_C \frac{\cos z}{\sin z} dz = \frac{1}{2\pi i} \cdot 2\pi i \left\{ \mathrm{Res}\left(\frac{\cos z}{\sin z}; \pi\right) + \mathrm{Res}\left(\frac{\cos z}{\sin z}; 0\right) + \mathrm{Res}\left(\frac{\cos z}{\sin z}; -\pi\right) \right\}$$

$$= 3.$$

Alternatively, note that, for $f(z) = \sin z$: holomorphic

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = \# \text{ of zeros of } f \text{ in } C, \text{ by argument principle.}$$

$$= 3.$$

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4. Consider a complex vector space

$$V := \{a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 | a_1, a_2, a_3, a_4 \in \mathbb{C}\}$$

where $f_1(z) = z$, $f_2(z) = \overline{z}$, $f_3(z) = e^z$, $f_4(z) = e^{\overline{z}}$ are complex-valued functions. Here $\overline{z}$ is the complex conjugate of z. For the unit circle $C : |z| = 1$ oriented in counter-clockwise manner in the complex plane $\mathbb{C}$, define a map $T : V \to \mathbb{C}$ by

$$T(f) = \int_C f(z)dz.$$

Prove that T is a linear map. Find a basis of $\ker(T)$ of the linear map T, and using $\ker(T)$, express $T^{-1}(2) = \{f \in V | T(f) = 2\}$. [20 points]

The map T is linear by the linearity of line integral.

Observe that

$T(f_1) = T(f_3) = 0$ $\because$ f_1, f_3: holomorphic in a simply connected region containing C. Cauchy's thm applies.

From $|z| = 1$, $\bar{z} = \frac{1}{z}$.

$T(f_2) = \int_C \frac{1}{z} dz = 2\pi i$

$T(f_4) = \int_C e^{1/z} dz = 2\pi i$.

Hence $f_2 - f_4 \in \ker T$.

$a f_1 + b f_3 + c(f_2 - f_4) = 0$

$\Leftrightarrow az + b \cdot \frac{1}{z} + ce^z - ce^{1/z} = 0 \ \forall z$

$\Rightarrow a = 0, b = 0, c = 0$

Therefore $\ker T = \text{Span}\{f_1, f_3, (f_2 - f_4)\}$

A basis of $\ker T$ is thus $\{f_1, f_3, (f_2 - f_4)\}$.

Now $T(f) = 2$

$\Leftrightarrow T(\sum_{i=1}^{4} a_i f_i) = 2$

$\Leftrightarrow a_2 T(f_2) + a_4 T(f_4) = 2$

$\Leftrightarrow 2\pi i(a_2 + a_4) = 2$

$\Leftrightarrow a_4 = \frac{1}{\pi i} - a_2$

So $T^{-1}(2) = \{a_1 f_1 + a_3 f_3 + a_2(f_2 - f_4) + \frac{1}{\pi i} f_4 : a_1, a_2, a_3 \in \mathbb{C}\}$

$= \ker T + \frac{f_4}{\pi i}$

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5. Show that the only automorphism of the unit disc with $f(0) = 0$, $f'(0) > 0$ is the identity map $f(z) \equiv z$. [10 points]

The only automorphism of $\mathbb{D}$ Satisfying $f(0)=0$ is a rotation: $f(z) = e^{i\theta} z$. For $f'(0) = e^{i\theta} > 0$, θ Must be $2n\pi$, $n \in \mathbb{Z}$. $\therefore$ $f(z) \equiv z$. ✓

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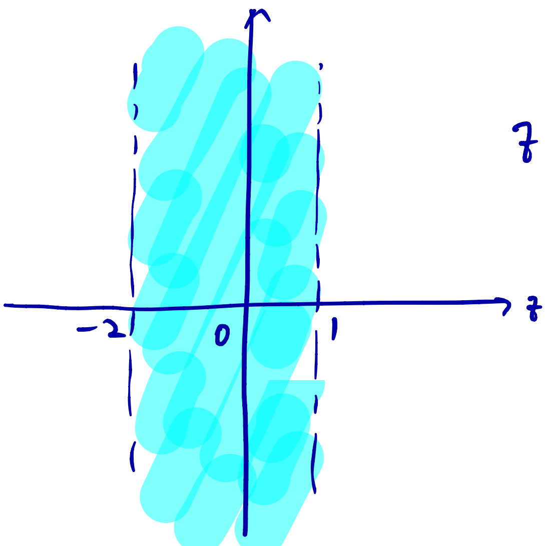
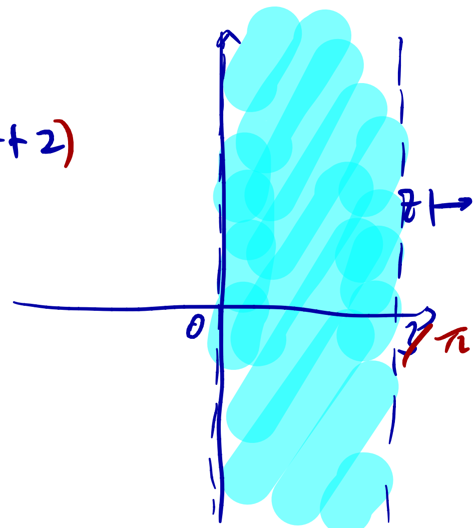
6. Find a conformal mapping from $S = \{z \in \mathbb{C} : -2 < \mathrm{Re}(z) < 1\}$ to the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ [10 points]



$z \mapsto \frac{\pi}{3}(z+2)$



$z \mapsto e^{iz}$ $\longrightarrow \mathbb{H}$

$\downarrow \frac{z - i}{z + i}$

$\mathbb{D}$

So $z \mapsto \dfrac{e^{i\frac{\pi}{3}(z+2)} - i}{e^{i\frac{\pi}{3}(z+2)} + i}$ does the job. ✓

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7. Consider a Möbius transformation T defined on the extended complex plane $\mathbb{C} \cup \infty$ satisfying

$$T(0) = 2, \quad T(1) = 2i, \quad T(\infty) = -2.$$

Compute $T(2i)$. [10 points]

$-2 \quad 2 \quad 2i$

The Möbius transformation that takes z_1, z_2, z_3 to $\infty, 0, 1$ is given by $T^{-1}(z) = \dfrac{z - z_2}{z - z_1} \cdot \dfrac{z_3 - z_1}{z_3 - z_2}$.

Hence $T^{-1}(z) = \dfrac{z-2}{z+2} \cdot \dfrac{2+2i}{-2+2i}$ so we want z s.t. $T^{-1}(z) = 2i$.

$= 2i$

$$\frac{z-2}{z+2} = \frac{2i(-2+2i)}{2+2i} = \frac{2i(-1+i)}{1+i} = \frac{2i(-1+i)(1-i)}{2} = \frac{-2i}{2} \cdot -2i = -2$$

$\Leftrightarrow \quad z - 2 + 2(z+2) = 0$

$\Leftrightarrow \quad z = \dfrac{-2}{3}$

$\therefore T(2i) = -\dfrac{2}{3}$. ✓

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8. Derive the fundamental theorem of algebra as a corollary of Rouché's theorem. [15 points]

Consider $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_i \in \mathbb{C}$, $a_n \neq 0$.

$$\frac{P(z)}{a_n z^n} = 1 + \frac{1}{a_n}\left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right)$$

For $|z| > 1$

$$\left|\frac{P(z)}{a_n z^n} - 1\right| = \left|\frac{1}{a_n}\left(\frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z}\right)\right|$$

$$\leq \frac{1}{|z||a_n|}\left(|a_0| + \cdots + |a_{n-1}|\right) \overset{(*)}{<} 1$$

So we consider $|z| > \dfrac{|a_0| + \cdots + |a_{n-1}|}{|a_n|} =: \alpha$ so that $(*)$ holds

and consider $C: |z| = R > \alpha$.

Then on C, $\left|\dfrac{P(z)}{a_n z^n} - 1\right| < 1 \iff |P(z) - a_n z^n| < |a_n z^n|$

whereas $Z(a_n z^n) = n$ inside C. Therefore by Rouché's thm,

$P(z) = (P(z) - a_n z^n) + a_n z^n$ has n roots inside C. ✓