Your Name:

Midterm Exam
Fall 2021, Complex Analysis II
Mathematics Education, Chungbuk National University
28.10.2021 15:00–16:40

Instructions: Please write your name on each page. If you want some portion of your writings on your answer sheet not to be graded, just cross it out. You are not allowed to use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

Extra sheets: Use the blank page on the back of each answer sheet as your scrap paper. Your work on blank pages will not be graded. Do not write your answers on those blank pages. If you need more space for writing down your answers, please ask for additional sheets.

1. Answer whether each of the following statements is true or false. No need to give reasons or details. **Just say true or false**. 2 points for each correct answer, 0 point for no answer, and -2 points for each incorrect answer.

(1) The winding number of a curve C at 0 that wraps the unit circle k times is $\frac{1}{2\pi i}\int_C \frac{1}{z}dz$. T

(2) The value of i^i is $e^{-\frac{\pi}{2}}$. F $\left(\because i^i = \exp(i \log i) = \exp(i(\frac{\pi}{2}i + 2\pi k i)) \right)$

(3) If D is a simply connected domain, then a holomorphic branch of complex logarithm is defined by $\int_{z_0}^{z} \frac{d\zeta}{\zeta} + \log z_0$ for some $z_0 \in D$. F (D should not contain 0)

(4) The winding number $n(\gamma, \alpha)$ of the curve γ at $\alpha \in \mathbb{C}$ is a constant in the connected components of the complement of γ. T

(5) The $\mathbb{C}$-valued function $f(z) = \frac{z}{e^z - 1}$ has a removable singularity at $z = 0$. T.

$$z f(z) = \frac{z^2}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots} . \quad \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots} = 0.$$

By Riemann's theorem on removable singularity, f has a removable singularity at $z=0$.

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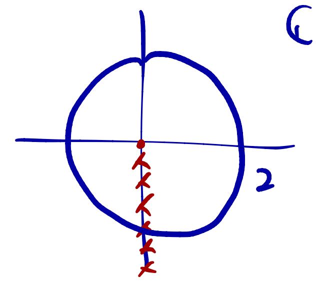
2. The principal value of the complex logarithm is defined on $-\frac{\pi}{2} \leq \operatorname{Arg}(z) < \frac{3\pi}{2}$ and $z^{1/2}$ is defined on that holomorphic branch of the logarithm. Compute $\int_{|z|=2} 3z^{1/2}dz$. [10 points]

Sol:

$$\int_{|z|=2} 3z^{1/2}\,dz = \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} 3e^{\frac{1}{2}\log 2e^{i\theta}}\, 2ie^{i\theta}\,d\theta$$



let $z = 2e^{i\theta}$

$dz = 2ie^{i\theta}d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} \Big(\underbrace{3e^{\frac{1}{2}\log 2 + \frac{1}{2}\log e^{i\theta}}}_{3\sqrt{2}\,e^{\frac{i\theta}{2}}}\Big)\, 2ie^{i\theta}\,d\theta$$

$$= 6\sqrt{2}i \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\frac{3i\theta}{2}}\,d\theta = \frac{6\sqrt{2}i}{3i/2}\, e^{\frac{3i\theta}{2}} \Big|_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= 4\sqrt{2}\left(e^{\frac{9}{4}\pi i} - e^{-\frac{3\pi}{4}i}\right)$$

$$= 4\sqrt{2}\left(e^{\frac{1}{4}\pi i} - e^{-\frac{3\pi}{4}i}\right)$$

$$= 4\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right)$$

$$= 8(1+i)$$ ✓

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3. Let $D = \{z \in \mathbb{C} | 0 < |z| < 1\}$ be a region in the complex plane $\mathbb{C}$ and a function $f\colon D \to \mathbb{C}$ be holomorphic. For an arbitrary $z \in D$, the function $f(z)$ satisfies an inequality

$$|f(z)| \le 1 + \ln\left(\frac{1+|z|}{2|z|}\right).$$

Show that $f(z)$ has a removable singularity at $z = 0$. If $f\left(\frac{1}{2}\right) = 1$, compute the value of $f\left(\frac{1+i}{3}\right)$. [15 points]

Sol: Note that $1+\ln\left(\frac{1+|z|}{2|z|}\right)$ for $0<|z|<1$ has a lower bound 1. Hence by Riemann's theorem for removable singularity, from $|z f(z)| \to 0$ as $|z| \to 0$ (so $\lim_{z\to 0} z f(z) = 0$), f has a removable singularity at $z=0$.

Now let $g(z) = \begin{cases} f(z) & \text{on } D \\ \lim_{z\to 0} f(z) & \text{if } z=0 \end{cases}$ be a holomorphic function defined on $|z|<1$. By the Maximum modulus principle, f has a maximum when $|z|=1$, whereas $\lim_{z\to 1^-} |f(z)| = 1$. Note that $f(\frac{1}{2}) = 1$. This means that f must be a constant. Hence $f\left(\frac{1+i}{3}\right) = 1$. ✓

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4. Let

$$f(x) = \frac{e^x - 1}{1 - x}$$

be a function defined on $\{x \in \mathbb{R} | -1 < x < 1\}$. Find the 3rd-order Taylor polynomial expanded at $x = 0$. Also compute the integral

$$\int_C \frac{e^z - 1}{z^4(1-z)} dz,$$

where C is a curve in the complex plane traveling along the circle centered at 0 with radius $\frac{1}{2}$ revolving once in counterclockwise direction. [10 points]

Sol. $e^x - 1 = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad |x|<1$

$f(x) = \frac{e^x - 1}{1-x} = x + \frac{3}{2}x^2 + \underbrace{\left(\frac{1}{6}+\frac{1}{2}+1\right)}_{\frac{5}{3}}x^3 + \cdots$

So the 3rd order Taylor polynomial is $\frac{5}{3}x^3 + \frac{3}{2}x^2 + x$.

$\frac{e^z - 1}{z^4(1-z)} = \frac{f(z)}{z^4} = \frac{1}{z^3} + \frac{3}{2}\frac{1}{z^2} + \frac{5}{3}\frac{1}{z} + \cdots$

So $\int_C \frac{e^z - 1}{z^4(1-z)} dz = 2\pi i \operatorname{Res}\left(\frac{e^z - 1}{z^4(1-z)}, 0\right) = 2\pi i \frac{5}{3} = \frac{10\pi i}{3}.$

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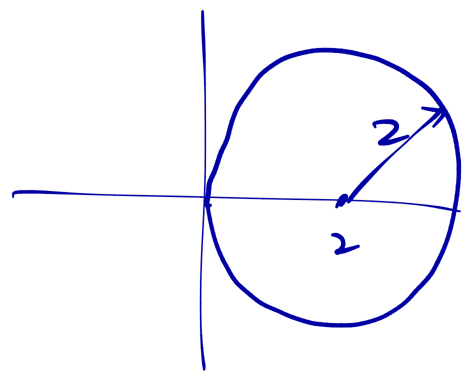
5. Show that an equation in complex variables $z + e^{-z} = 2$ has only one complex root in $|z-2| < 2$, and also show that the root is real. [20 points]

Sol: $f(z) = z-2$

$g(z) = e^{-z}$.

$\gamma = \{ z \in \mathbb{C} : |z-2| = 2 \}$



On γ, $|f| = 2$.

$e^{-4} \leq |g| \underset{z=x+iy}{=} e^{-x} \leq 1$

So $|f| > |g|$, whereas f has only one root insider namely $z=2$. Therefore, by Rouche's theorem, $f+g = z-2+e^{-z} = 0$ has a unique root in $|z-2| < 2$.

To see the root of $f+g$ is real, consider $h(x) = x-2+e^{-x}$. Clearly $h(1) = -1+e^{-1} < 0$, $h(3) = 1+e^{-3} > 0$. Since $h(x)$ is continuous on $\mathbb{R}$, by the intermediate value theorem, h has a root in $(0,4)$.

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6. Suppose f is holomorphic inside and on a positively oriented simple closed curve γ and has no zeros on γ. Show that if m is a positive integer then

$$\frac{1}{2\pi i}\int_{\gamma} z^m \frac{f'(z)}{f(z)}dz = \sum_k (z_k)^m$$

where z_k is the k^{th} zero of f inside γ. [10 points]

Sol: By the Residue theorem,

$$\frac{1}{2\pi i}\int_{\gamma} z^m \frac{f'}{f} dz = \frac{\cancel{2\pi i}}{\cancel{2\pi i}} \sum_k \operatorname{Res}\left(z^m \frac{f'}{f} ; z_k\right)$$

$$= \sum_k \left.\frac{z^m f'}{f'}\right|_{z=z_k} = \sum_k (z_k)^m \checkmark$$

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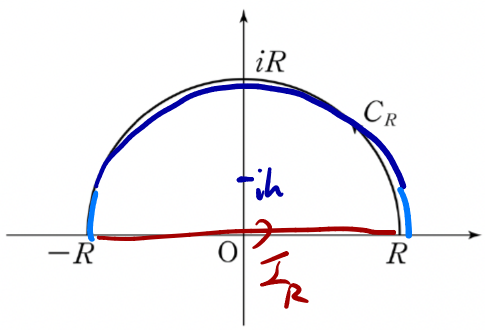
7. Consider $C_R = \{Re^{it} \in \mathbb{C} | 0 \le t \le \pi\}$ a semicircle with radius R in the complex plane $\mathbb{C}$ as in the figure below. For $a > 0$ and $b > 0$, show that

$$\lim_{R\to\infty} \int_{C_R} \frac{ze^{ibz}}{z^2+a^2} dz = 0$$

and compute

$$\int_{-\infty}^{\infty} \frac{xe^{ibx}}{x^2+a^2} dx.$$



[15 points]

Sol: Let $z \in C_R$. Then $z = Re^{it}$, $t \in [0, \pi]$.

If we let $z = x+iy$, $ibz = -by + ibx$.

Let $A = \{ z \in C_R : \mathrm{Im}\, z \geq h \}$

$B = \{ z \in C_R : \mathrm{Im}\, z < h \}$

$$\left| \int_A \frac{ze^{ibz}}{z^2+a^2} dz \right| \le \frac{Re^{-bh}}{R^2+a^2} \cdot \pi R = C_1 e^{-bh}.$$

$$\left| \int_B \frac{ze^{ibz}}{z^2+a^2} dz \right| \le \frac{R}{R^2+a^2} \cdot 4h = C_2 \frac{h}{R}.$$

Set $h = \sqrt{R}$

Then $\left| \int_{C_R} \frac{ze^{ibz}}{z^2+a^2} dz \right| \le C_1 e^{-b\sqrt{R}} + C_2 \frac{\sqrt{R}}{R} \to 0$ as $R \to \infty$.

$\therefore \int_{-\infty}^{\infty} \frac{xe^{ibx}}{x^2+a^2} = \int_{C_R+I_R} \frac{ze^{ibz}}{z^2+a^2} dz = 2\pi i \,\mathrm{Res}\left(\frac{ze^{ibz}}{z^2+a^2}; ai \right) = 2\pi i \frac{e^{-ab}}{2} = e^{-ab}\pi i$

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8. Compute

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

by using the residue theorem. (Cf. $\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \frac{z^7}{4725} + O(z^9)$. No need to justify how you have obtained the given infinite sum as a residue of a some complex-valued function at a point. The grader will look at only the final answer.) [10 points]

$\therefore \mathrm{Res}(\pi \cot \pi z; n) = 1$

Idea: $\int_{C_N} f(z)\, \pi \cot \pi z \, dz \xrightarrow{} 0 \quad = 2\pi i \left(\sum_{n=-N}^{N} f(n) + \sum_k \mathrm{Res}(f(z) \pi \cot \pi z; z_k) \right)$

where $\{z_k\}$ poles of f.

So $\sum_{\substack{-\infty \\ n \neq 0}}^{\infty} f(n) = - \sum_k \mathrm{Res}\left(f(z) \pi \cot \pi z ; z_k \right)$

In this problem, $f(z) = \frac{1}{z^4}$

$$\frac{\pi \cot \pi z}{z^4} = \frac{\pi}{\pi z^5} - \frac{\pi^2 \cancel{z}}{3z^3} - \frac{\pi^4 z^3}{45 z} - \cdots$$

So $-\mathrm{Res}\left(\frac{\pi \cot \pi z}{z^4} ; 0 \right) = \frac{\pi^4}{45}$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2} \sum_{-\infty}^{\infty} \frac{1}{n^4} = \underline{\underline{\frac{\pi^4}{90}}}$.