Your Name:

Final Exam
Spring 2021, Complex Analysis I
Mathematics Education, Chungbuk National University
10.06.2021 10:00–11:40

Instructions: Please write your name on each page. If you want some portion of your writings on your answer sheet not to be graded, just cross it out. You are not allowed to use your textbook or notes. You cannot use any electronic device in this exam. You are not allowed to talk to other students. Please write all details explicitly. Answers without justifications and/or calculation steps may receive no score.

Extra sheets: Use the blank page on the back of each answer sheet as your scrap paper. Your work on blank pages will not be graded. Do not write your answers on those blank pages. If you need more space for writing down your answers, please ask for additional sheets.

1. Answer whether each of the following statements is true or false. No need to give reasons or details. **Just say true or false.** 2 points for each correct answer, 0 point for no answer, and −2 points for each incorrect answer.

(1) The radius of convergence of the series $\sum_{n=1}^{\infty}(z_n - 3i)^n z^n$ ($z \in \mathbb{C}$) for the sequence $z_n = \frac{2ni^n}{n+1}$ ($n = 1, 2, \cdots$) is 1/5 True

(2) The stereographic projection $S^2 \to \mathbb{C} \cup \{\infty\}$ takes circles to circles. True

(3) If a function $f(x+iy) = u(x,y) + iv(x,y)$ satisfies Cauchy-Riemann equations, then f is holomorphic at $x+iy$. False

(4) The Cauchy-Goursat theorem states that for a simple closed curve C in a simply connected region and a function holomorphic on C, the contour integral $\int_C f(z)dz$ is always zero. False

(5) Let $\{f_n\}$ be a sequence of functions holomorphic in a region D and such that $f_n \to f$ uniformly. Then f is holomorphic in D. True

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2. Show that a nonconstant holomorphic function cannot map a region into a straight line or into a circular arc. [10 points]

Let f be a nonconstant holomorphic function.
Suppose that f maps a region into a circular arc or into a Straight line. We may assume that the circle is Centered at the origin and the Straight line is $\mathrm{Re}\, z = 1$ origin because a translation in $\mathbb{C}$ is nonconstant & holomorphic.
and a rotation

Then for the circular arc image $|f| = \text{constant} \Rightarrow f$: constant *

for the Straight line image for holomorphic $\Rightarrow f$: const
$f = u + iv$, u: const *

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3. (1) State and prove Liouville's theorem.[5 points]

(2) Prove that the image of a nonconstant entire function is dense in $\mathbb{C}$. [10 points]

(1) Every bounded entire function is constant.

pf Suppose $|f| \le M$

$$|f'(z)| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\right| \le \frac{1}{2\pi}\,\frac{M}{R^2}\cdot 2\pi R = \frac{M}{R}$$

Since R: arbitrary, $f'(z)=0$ $\Rightarrow f$: const.

(2) Suppose f: nonconstant entire, $\exists\,\delta$ s.t. $D(z_0;\delta)\cap \operatorname{Im} f = \phi$.

Then $g(z) = \dfrac{1}{f(z)-z_0}$ is holomorphic everywhere.

Since $|f(z)-z_0| \ge \delta$ for all z, $|g(z)| \le \frac{1}{\delta}$. $\therefore$ by Liouville's thm, g: constant. Then f: const $\star$ /

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4. Prove the fundamental theorem of algebra using the minimum modulus theorem. [15 points]

Let $p(z)$ be a polynomial. By the minimum modulus thm, $p(z)$ does not attain a minimum unless $p(z)=0$ whereas $p(z)$ is an entire function attaining the minimum modulus when $z=0$. Therefore $p(z)$ must have a solution in $\mathbb{C}$. i.e. every nonconstant polynomial with $\mathbb{C}$-coeff has a zero in $\mathbb{C}$. ✓

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5. Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$. [10 points]

By the uniqueness theorem, there is only one entire function satisfying $f(x) = e^x \ \forall x \in \mathbb{R}$

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6. Show that the function $f(z) = \int_0^1 \frac{\sin zt}{t} dt$ is an entire function. [15 points]

Take any rectangle Γ in $\mathbb{C}$. Clearly f is continuous

and $\int_\Gamma f(z)\,dz = \int_0^1 \frac{1}{t} \underbrace{\int_\Gamma \sin zt \, dz}_{=0} \, dt = 0$. Here $z \in \mathbb{C}$: arbitr

Therefore by Morera's thm, f: entire. /

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7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A complex number $w \in \mathbb{D}$ is a fixed point for the map $f : \mathbb{D} \to \mathbb{D}$ if $f(w) = w$. Prove that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and has two distinct fixed points, then f is the identity, that is $f(z) = z$ for all $z \in \mathbb{D}$. [15 points]

Let $f(z_i) = z_i \quad i = 1, 2 \quad , z_i \in \mathbb{D}$

Consider $B_{z_1}(z) = \dfrac{z_1 - z}{1 - \bar{z}_1 z}$ Sending 0 to z_1, z_1 to 0

$g = B_{z_1} \circ f \circ B_{z_1}$ takes 0 to 0 and $|g| \leq 1$.

By Schwarz lemma $|g(z)| \leq |z|$

Now $\hat{z} = B_{z_1}(z_2)$ is a nonzero fixed point of g, ($\because B_{z_1} \circ B_{z_1} = \mathbb{1}$)

From the proof of the Schwarz lemma $\dfrac{|g(z)|}{|z|} = 1$ for some z would mean by Maximum modulus principle $\dfrac{g(z)}{z}$ has to be a constant. From $g(\hat{z}) = e^{i\theta}\hat{z}$, $\theta = 2n\pi$, $n \in \mathbb{Z}$

Therefore $g = id$, $\Rightarrow f = id$. ✓

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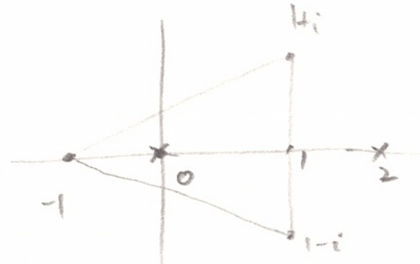
8. Let C be a triangle in $\mathbb{C}$ whose vertices are -1, $1 - i$, and $1 + i$. Calculate the integral

$$\int_C \frac{dz}{z(z-2)}.$$

[10 points]



$$\int \frac{dz}{z(z-2)} = \int \frac{\frac{1}{z-2}\,dz}{z} = 2\pi i \left.\frac{1}{z-2}\right|_{z=0} = -\pi i$$

↑ Cauchy integral formula + General Cauchy's thm